Non-negativity Analysis for Exponential-Polynomial-Trigonometric Functions on the non-negative real half-line

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Abstract. The class of real exponential-polynomial-trigonometric (EPT) functions is ubiquitous in the mathematical sciences. It is the class of functions that appear as solutions to linear differential equations with constant real coefficients. In linear dynamical systems theory they appear as impulse response functions and in that context are often represented as $y(t) = ce^{At}b$, where c is a row vector, A a square matrix and b a column vector. The name EPT functions draws on the fact that these functions can be written in the form $\sum_{i=0}^{d} q_i(t)e^{\lambda_i t}\cos(\theta_i t+\tau_i)$, where the q_i are real polynomials and λ_i, θ_i and τ_i are real numbers. These functions also appear in probability theory (as density and distribution functions) and financial mathematics (e.g. as forward rate curves). In such applications their non-negativity is often required. In the present paper we address the question of how to characterize non-negative EPT functions on the non-negative real half-line, using earlier results obtained for finite intervals. We describe necessary conditions and we present a new sufficient condition and methods for verification of this sufficient condition. The main idea is to represent an EPT function as the product of a row vector $b(t)$ of EP functions and a column vector $F(t) := f(e^{i\theta_1 t}, \ldots, e^{i\theta_m t})$ of multivariate trigonometric polynomials with unimodular exponential functions $e^{i\theta_k t},\ k = 1, 2, \ldots, m$ as arguments; here $\theta_k,\ k = 1, 2, \ldots, m$ are chosen to be real numbers which are linearly independent over the set of rational numbers $\mathbf{Q}$. From the theory of almost periodic functions it follows that the closure of the set of vectors $F(t)$ obtained by varying t over the interval $[T, \infty)$ for any $T > 0$, is equal to the set $f(\mathbf{T}^m)$, where $\mathbf{T}^m$ denotes the $m-$dimensional unit torus in $\mathbf{C^m}$. This will be used to describe a necessary condition and a sufficient condition. A method for verifying whether the sufficient condition is satisfied is presented, based on minimization over the torus of a continuous level function which is non-negative iff the sufficient condition is satisfied. The methods are based on the availability of a generalized Budan-Fourier sequence technique to determine the minimum of an EPT function on a given finite interval $[0, T]$ which is presented elsewhere by the authors.

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1. Introduction

We consider a class of functions on $[0, \infty)$ that can be described in various ways:

- The *Euler-d'Alembert*[1] class of infinitely differentiable functions $y : [0, \infty) \to \mathbf{R}$ that satisfy a real linear differential equation with constant coefficients:

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_n y = 0,$$

[1] cf. Euler (1743), d'Alembert (1748)

with real initial conditions:

$$y(0) = b_1, y^{(1)}(0) = b_2, \dots, y^{(n-1)}(0) = b_n$$

- The *matrix-exponential* class of functions of the form

$$y(t) = ce^{At}b, \ A \in \mathbf{R}^{n\times n}, c \in \mathbf{R}^{1\times n}, b \in \mathbf{R}^{n\times 1}$$

with $t \geq 0$, where A, b, c are independent of t.
- The class of real exponential-polynomial-trigonometric (EPT) functions $y : [0,\infty) \to \mathbf{R}$ of the form

$$y(t) = \Re\left(\sum_{k=1}^{K} p_k(t)e^{\mu_k t} =\right) = \frac{1}{2}\sum_{k=1}^{K} p_k(t)e^{\mu_k t} + \frac{1}{2}\sum_{k=1}^{K} \overline{p_k}(t)e^{\overline{\mu_k}t},$$

where $p_k(t) \in \mathbf{C}[t], \ \mu_k \in \mathbf{C}, k = 1, 2, \dots, K, \ t \geq 0$ and the bar denotes complex conjugation. Note that this can also be written in the form $y(t) = \sum_{i=0}^{d} q_i(t)e^{\lambda_i t}\cos(\theta_i t + \tau_i)$, where the q_i are real polynomials and λ_i, θ_i and τ_i are real numbers and $K \leq d \leq 2K$.

The fact that these classes are equal is well-known. The first class is contained in the second one as can be seen using the companion matrix with characteristic polynomial $z^n + a_1 z^{n-1} + \ldots + a_n$; the second one is in the first as can be seen using the theorem of Cayley-Hamilton; the second one is contained in the third as follows from the theory of the Jordan normal form; that EPT functions satisfy a linear differential equation with constant coefficients follows from the fact that $(\frac{d}{dt} - \lambda)^{k+1}(t^k e^{\lambda t}) = 0$ and the fact that the Euler-d'Alembert class forms a ring of functions.

A further useful characterization of this class can be given as the class of continuous functions on $[0,\infty)$ for which the Laplace transform exists and is rational and strictly proper. In fact the Laplace transform is equal to $r(z) = c(zI - A)^{-1}b$ as is well-known (for instance in linear systems theory where r is called the transfer function and the corresponding EPT function is called the impulse response function). For any strictly proper rational function r one can construct a triple (A, b, c) such that $r(z) = c(zI - A)^{-1}b$ (in system theory the collection of methods that allow one to find such triples for a given rational function is called realization theory), and the corresponding EPT function y is then given by $y(t) = ce^{At}b$.

Important subclasses can be characterized by the location of the poles of the Laplace transform (or transfer function). These poles are the eigenvalues of the matrix A in a so-called minimal representation (or realization) (A, b, c) of an EPT function (for the theory of minimal representation or realization one can refer to the theory of linear state space systems, cf. e.g., [3] or Chapter 10 of [4] for a more algebraic approach). The subclass P of polynomials coincides with the subclass for which all eigenvalues of A are zero. The subclass E of *real exponential sums*, i.e., linear combinations with constant coefficients of real exponential functions of the form $e^{\lambda t}$ coincides with the subclass for which the eigenvalues (resp., poles) are real and distinct. The subclass T of trigonometric functions is the subclass for which the eigenvalues (resp., poles) are purely imaginary and distinct.

Clearly these functions are ubiquitous in the mathematical sciences! Here we want to mention some examples in which these functions appear in such a way that they are required to be non-negative.

- In *financial mathematics* they appear as *forward rate curves,* e.g. the Nelson-Siegel forward rate curves (cf [5]):

$$z_0 + z_1 e^{-\lambda t} + z_2 t e^{-\lambda t}$$

  or the Svensson forward rate curves (cf. [6]). As the function values denote interest rates we want them to be non-negative!
- In *probability theory* they appear as *probability density functions.* In the form of Gamma densities with positive integer shape parameter k for instance: $f(t;k,\beta) = \frac{\beta^k}{(k-1)!} t^{k-1} e^{-t\beta},\ t \geq 0,\ k \in \mathbf{N}, \beta > 0$. Such density functions must be non-negative and integrable to one.
- In *system theory* they appear as impulse response functions of linear systems. In the case of so-called positive (resp., non-negative) systems one wants the impulse response functions to be positive (resp., non-negative). (Various definitions for positive and non-negative systems can be found in the literature, however this is a minimal requirement).

Given the importance of non-negative EPT functions in these and other areas, the question arises how to analyze whether a given EPT function is non-negative or not and how to characterize classes of non-negative EPT functions. The present paper addresses these questions. Use will be made of an earlier method that was found by the authors, to determine whether an EPT function is non-negative on any given finite interval $[0, T]$ where $T > 0$ is an arbitrary real positive number. In the next section a summary of that method will be given as it will be used in the sequel. The main results of this paper are (i) a necessary condition as well as (ii) a sufficient condition for non-negativity of the tail of an EPT function; (iii) a method that can be used to determine whether the sufficient condition is satisfied.

2. NON-NEGATIVITY ANALYSIS ON A FINITE INTERVAL (SUMMARY)

In [9] a method is presented to determine non-negativity of an EPT function $y(t)$ on a given finite interval $t \in [0, T]$. The method is based on the construction of a generalized Budan-Fourier (BF) sequence. In case $\mu_k \in \mathbf{R},\ k = 1, 2, \ldots, K$, we speak of an exponential-polynomial (EP) function. It can be represented by $y(t) = ce^{At}b$, for some triple $(A, b, c) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times 1} \times \mathbf{R}^{1\times n}$, where the eigenvalues of A are real and can be ordered in decreasing order as $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. For such an EP function $y(t)$ a BF sequence is given by:

$$
\begin{aligned}
y_0(t) &:= y(t) = ce^{At}b \\
y_1(t) &:= c(\lambda_1 I - A)e^{At}b \\
y_2(t) &:= c(\lambda_1 I - A)(\lambda_2 I - A)e^{At}b \\
&\vdots \\
y_n(t) &:= c(\lambda_1 I - A)(\lambda_2 I - A)\dots(\lambda_n I - A)e^{At}b \equiv 0
\end{aligned}
$$

It has the property that $y_k(t)$ has at most one sign-changing zero in between any two consecutive sign-changing zeros or boundary points of $y_{k+1}(t),\ k = 0, 1, \dots, n-1$ on $[0, T]$. Using this and applying a bisection technique we can find all sign-changing zeros of $y_n(t), y_{n-1}(t), \dots, y_0(t)$ and hence determine whether or not $y_0(t) \equiv y(t)$ is non-negative. In [9] a BF sequence is also presented for any EPT function $y(t)$ on $[0, T]$. This allows us to determine whether $y(t) \geq 0,\ \forall t \in [0, T]$. Based on this we now want to study the possible tail behaviour of non-negative EPT functions. Obviously if we can show that an EPT function $y(t)$ is non-negative for all $t \geq T_0$ for some $T_0 > 0$ and we can verify that $y(t)$ is non-negative on $[0, T_0]$ using the BF sequence method, then non-negativity of $y(t)$ on $[0, \infty)$ follows.

3. A SPECIAL REPRESENTATION OF EPT FUNCTIONS

The following representation of an arbitrary EPT function will play an important role for the remainder of the paper.

Theorem 3.1. *Any EPT function can be written in the form*

$$y(t) = \sum_{k=0}^{N} b_k(t)\Re(f_k(e^{i\theta_1 t}, e^{i\theta_2 t}, \dots, e^{i\theta_m t})),$$

where each $b_k(t)$ is an EP function of the form

$$b_k(t) = (t + T_1)^{d_k} e^{\lambda_k(t+T_1)},\ k = 0, 1, \dots, N,$$

for some $T_1 \geq 0$, such that $b_0(t) > b_1(t) > \dots > b_N(t),\ \forall t > 0$ and $f_k(z_1, z_2, \dots, z_m)$, $|z_j| = 1,\ j = 1, 2, \dots, m$ is a multivariate trigonometric polynomial, $k = 0, 1, \dots, N$, and $\theta_1, \theta_2, \dots, \theta_m \in \mathbf{R}$ are linearly independent over $\mathbf{Q}$.

Remark. A real-valued function f on the $m-$dimensional unit torus $\mathbf{T}^m = \{z = (z_1, z_2, \dots, z_m) : |z_j| = 1,\ j = 1, 2, \dots, m\}$, will be defined to be a trigonometric polynomial if it is of the form

$$f = \sum_{\alpha \in I^m} c_\alpha z^\alpha$$

where $c_\alpha \in \mathbf{C}$ for each $\alpha = (\alpha(1), \alpha(2), \dots, \alpha(m)) \in I^m$, $z^\alpha := z_1^{\alpha(1)} z_2^{\alpha(2)} \dots . z_m^{\alpha(m)}$ and I is a finite subset of the integers $\mathbf{Z}$. So, in particular, negative powers are allowed. Note that $\Re f(z_1, z_2, \ldots, z_m) = \frac{1}{2} f(z_1, z_2, \ldots, z_m) + \frac{1}{2} \bar{f}(z_1^{-1}, z_2^{-1}, \ldots, z_m^{-1})$ where $\bar{f}(z_1, z_2, \ldots, z_m) = \sum_{\alpha \in I^m} \overline{c_\alpha} z^\alpha$ and $\overline{c_\alpha}$ is the complex conjugate of c_α.

Proof. From the representation of a *complex* EPT function as a sum of products of polynomials with complex exponential functions, it follows, by treating each of the monomials in the polynomials separately, that an EPT function $y(t)$ can be written as

$$y(t) = \Re \left(\sum_{k=0}^{N} \tilde{b}_k(t) \tilde{l}_k(e^{i\eta_1 t}, e^{i\eta_2 t}, \ldots, e^{i\eta_K t}) \right)$$

where each $\tilde{b}_k(t)$ is an EP function of the form

$$\tilde{b}_k(t) = t^{d_k} e^{\lambda_k t}, \ k = 0, 1, \ldots, N,$$

such that $\tilde{b}_0(t) > \tilde{b}_1(t) > \ldots > \tilde{b}_N(t), \ \forall t > T_1$ for some sufficiently large number $T_1 > 0$; and $\tilde{l}_k(w_1, w_2, \ldots, w_K)$ is a degree-one (linear) polynomial defined on the $K-$dimensional unit torus for each $k = 0, 1, 2...., N$. Here the λ_k are the real parts of the eigenvalues of the matrix A and η_j are the imaginary parts of the eigenvalues of A, where A is the square matrix in a minimal (A, b, c) representation of the EPT function (i.e. $y(t) = ce^{At}b$). Note that because of the nature of EP functions it follows that $\lim_{t\to\infty} \frac{\tilde{b}_{k+1}(t)}{\tilde{b}_k(t)} = 0$ for each $k = 0, 1, 2, \ldots, K$ hence $\tilde{b}_k(t) = o(\tilde{b}_0(t))$ for $t \to \infty$ for each $k = 1, 2, \ldots, K$. Now consider the auxiliary function $g(t) := ce^{-AT_1} e^{At} b$. Note that $g(t + T_1) = y(t)$ for all $t \geq 0$. The function g is EPT and hence it has a representation of the form given above (note that g and y can be represented using the *same* matrix A; hence the spectrum is the same in both cases):

$$g(t) = \Re \left(\sum_{k=0}^{N} \tilde{b}_k(t) \hat{l}_k(e^{i\eta_1 t}, e^{i\eta_2 t}, \ldots, e^{i\eta_K t}) \right),$$

with $\hat{l}_k$ again a linear polynomial on the $K-$dimensional torus, for each $k = 0, 1, \ldots, N$. It follows from $y(t) = g(t + T_1)$ that

$$y(t) = \Re \left(\sum_{k=0}^{N} b_k(t) l_k(e^{i\eta_1 t}, e^{i\eta_2 t}, \ldots, e^{i\eta_K t}) \right),$$

where l_k is another linear polynomial on the $K-$dimensional torus, for each $k = 0, 1, \ldots, N$. Now consider the vector space V over the field of rational numbers $\mathbf{Q}$ spanned by the real numbers $\eta_1, \eta_2, \ldots, \eta_K$. Let m be the dimension of this vector space V and consider a $V-$basis $\theta_1, \theta_2, \ldots, \theta_m$ which has the special property that each of the elements $\eta_1, \eta_2, \ldots, \eta_K$ can be expressed as a linear combination of the basis elements with *integer* coefficients. Such a basis can be obtained from an arbitrary basis by an appropriate basis transformation: suppose that $\eta = \tilde{M}\tilde{\theta}$, where $\eta = (\eta_1, \eta_2, \ldots, \eta_K)'$, $\tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_m)'$ is some arbitrary $V-$basis and $\tilde{M} \in \mathbf{Q}^{K\times m}$. Let $\delta_i \in \mathbf{N}$ denote the least common denominator of the $i-$th column of $\tilde{M}$, for each $i = 1, 2, \ldots, m$. Then take $\theta_j := \frac{\tilde{\theta}_j}{\delta_j}$ and

D the diagonal matrix with δ_i as its $i-$th diagonal element. Then $M = \tilde{M}.D$ is integer-valued and $\theta_1, \theta_2, \ldots, \theta_m$ a basis with the required property. Let the $(i,j)-$th element of M be denoted by $m_{i,j} \in \mathbf{Z}$. We can now replace each η_j in the expression $l_k(e^{i\eta_1 t}, e^{i\eta_2 t}, \ldots, e^{i\eta_K t})$ by $\eta_j = \sum_{i=1}^{m} m_{j,i}\theta_i$. Once that is done for each η_j then we obtain a trigonometric polynomial f_k on the unit torus with $f_k(e^{i\theta_1 t}, e^{i\theta_2 t}, \ldots, e^{i\theta_m t}) = l_k(e^{i\eta_1 t}, e^{i\eta_2 t}, \ldots, e^{i\eta_K t})$. If we denote by m_j the multi-index given by the $j-$th row of M then we can write

$$f_k(z) = l_k(z^{m_1}, z^{m_2}, \ldots, z^{m_K}),\ k = 1, 2, \ldots, N.$$

By construction the numbers $\theta_1, \theta_2, \ldots, \theta_m$ are linearly independent over $\mathbf{Q}$. □

Let us say that an EPT function $y(t)$ is non-negative eventually if there exists a value T such that $y(t) \geq 0$ for all $t \geq T$. So, if an EPT function is non-negative eventually, hence non-negative on $[T, \infty)$, then one can apply the BF sequence approach to determine whether the EPT function is non-negative on $[0, T)$ as well and hence on all of $[0, \infty)$.

Theorem 3.2. *Assume without loss of generality that $\Re(f_0)$ is not the zero function: $\Re(f_0) \not\equiv 0$. Necessary conditions for an EPT function to be non-negative eventually are*

(i) *that $\Re(f_0(z_1, z_2, \ldots, z_m)) \geq 0,\ \forall z \in \mathbf{T}^m = \{z = (z_1, z_2, \ldots, z_m) : |z_k| = 1,\ k = 1, 2, \ldots, m\}$,*

(ii) *and hence that $\Re f_0$ has a positive constant term, and $\sigma(A)$ has a real dominant eigenvalue: $\lambda_0 \in \sigma(A)$ and all elements of $\sigma(A)$ have real part less than or equal to λ_0.*

Proof. ad (i) For any number $T > 0$ the set $\{(f_0(e^{i\theta_1 t}, \ldots, e^{i\theta_m t}), \ldots, f_N(e^{i\theta_1 t}, \ldots, e^{i\theta_m t}))|t \geq T\}$ has closure $\{(f_0(z), f_1(z), \ldots, f_N(z))|z \in \mathbf{T}^m\}$. This follows from the fact that $\theta_1, \theta_2, \ldots, \theta_m$ are linearly independent over $\mathbf{Q}$ and Kronecker's approximation theorem [7]. This implies that if there exists a point $z \in \mathbf{T}^m$ with the property that $\Re\left(f_0(z)\right) =: -\epsilon < 0$, then due to continuity of f_0 for each $T > 0$ there will be a number $t > T$ such that $\Re f_0(e^{i\theta_1 t}, \ldots, e^{i\theta_m t}) < -\epsilon/2$. Because the values that $f_1, f_2, \ldots, f_N$ can take on the torus are bounded (as $f_1, f_2, \ldots, f_N$ are continuous functions on a compact set) it follows that as functions of t, the functions $b_k(t) f_k(e^{i\theta_1 t}, e^{i\theta_2 t}, \ldots, e^{i\theta_m t}),\ k = 1, 2, \ldots, N$ are all $o\left(b_0(t)\right)$. It follows that the EPT function does have negative values for some $t > T$, for any positive number T. It follows that necessary for eventual non-negativity of $y(t)$ is that $\Re f_0(z) \geq 0$ for all z on the unit torus $\mathbf{T}^m$.

ad(ii) As is well-known the constant term of any trigonometric polynomial in m variables can be obtained by integrating it over the unit torus (viewed as a subset of Euclidean space) and dividing by $(2\pi)^m$. Applying this to the non-negative and not identically zero function $\frac{1}{2} f_0(z_1, z_2, \ldots, z_m) + \frac{1}{2}\bar{f}_0(z_1^{-1}, z_2^{-1}, \ldots, z_m^{-1})$ we obtain that the real part of the constant term of f_0 is positive. This implies that also the real part of the constant term of the corresponding linear polynomial l_0 is positive (it is the same, only the non-constant terms can change when going from l_0 to f_0) and hence that λ_0 is in the spectrum of A, where $y(t) = ce^{At}b$, with multiplicity

$d_0 + 1$. That λ_0 is at least as big as the real part of the other eigenvalues follows from the fact that b_0 is at least as large as the $b_k,\ k = 1, 2, \ldots, N$. □

Remark. The fact that if $y(t) = ce^{At}b$ is eventually non-negative then A must have a dominant real pole is also shown in [10], using the classical Pringsheim theorem about power series with nonnegative coefficients.

The question of determining criteria for the non-negativity of a trigonometrical polynomial on $\mathbf{T}^m$ arises naturally from this theorem. While the well-known L. Fejér and F. Riesz characterization of non-negative trigonometric polynomials of one real variable settles the issue when $m = 1$ [11], the question is more challenging when $m \geq 2$. However, it's easy to resolve it for a polynomial that is linear in each variable separately.

Theorem 3.3. *Let*

$$f(z) = c_0 + \sum_{k=1}^{m} c_k z_k,\ z = (z_1, z_2, \ldots, z_m) \in \mathbf{T}^m.$$

Then the real part of f is non-negative on $\mathbf{T}^m$ if and only if

$$\sum_{k=1}^{m} |c_k| \leq \Re c_0.$$

Proof. If the displayed condition holds, and $z \in \mathbf{T}^m$, then

$$\begin{aligned} \Re f(z) &= \Re c_0 + \sum_{k=1}^{m} \Re\{c_k z_k\} \\ &\geq \Re c_0 - \sum_{k=1}^{m} |c_k z_k| \\ &= \Re c_0 - \sum_{k=1}^{m} |c_k| \\ &\geq 0, \end{aligned}$$

whence the sufficiency part follows. Conversely, if $\Re f \geq 0$ on $\mathbf{T}^m$, and $1 \leq j \leq m$, consider c_j: if $c_j \neq 0$, select w_j so that $w_j c_j = -|c_j|$; and otherwise select $w_j = 1$. Then, $w = (w_1, w_2, \ldots, w_m) \in \mathbf{T}^m$ and so

$$0 \leq \Re f(w) = \Re c_0 - \sum_{k=1}^{m} |c_k|,$$

whence the necessity part follows. □

Theorem 3.4. *A sufficient condition for non-negativity of $y(t)$ on $[T, \infty),\ T \geq 0$ is*

$$\forall z \in \mathbf{T}^m : \forall t \in [T, \infty) : \sum_{k=0}^{N} b_k(t) \Re(f_k(z)) \geq 0$$

Proof. This is self-evident from the special representation result for an EPT function (see Theorem 3.1). □

Remarks.

- Whether there are non-negative EPT functions that do not satisfy this sufficient condition is not known to the authors at present, but there is a strong suspicion that such EPT functions will indeed exist. Still for many practical purposes the class of non-negative EPT functions satisfying this sufficient condition might be large enough. Further research in the properties of this sufficient condition is on-going.
- If one would use a representation of the type presented in Theorem 3.1 except that *not* all $\theta_1, \ldots, \theta_m$ would be linearly independent, this sufficient condition could still be used; however it would just not "catch" as many non-negative EPT functions as in the case of linear independence.
- A special case in which the sufficient condition for eventual non-negativity holds is when $\Re(f_0)$ has a positive minimum on the unit torus; the sufficient condition is satisfied in that case, for sufficiently large T, because $b_k(t) = o(b_0(t)),\ k = 1, 2, \ldots, N$, for $t \to \infty$.
- A further special case of this is obtained when the set of elements in the spectrum with maximum real part consists just of one point (higher multiplicity would be no problem in this case, only the location of the points matters here). According to Theorem 3.1 such a point in the spectrum would actually have to be real. Then $\Re f_0$ is a positive constant hence the sufficient condition is satisfied for sufficiently large T. The class of EPT functions satisfying this condition is likely to be very important in practice; for this class one can determine non-negativity by determining an appropriate T such that the function is non-negative for all $t > T$, while for $t \in [0, T]$ one can use the BF sequence approach mentioned in the previous section to determine non-negativity (assuming it is possible to determine the sign of the EPT functions involved in the BF sequence at each of the points at which we need to know it).
- And a further special case of this is when the function is actually an EP function, because then all eigenvalues are real hence also the one with largest real part.

4. A PROJECTION ON THE BOUNDARY OF A SET OF NONNEGATIVE FUNCTIONS

The question now arises as to how one can ascertain that a given EPT function satisfies the sufficient condition for some sufficiently large value of T. As was noted in the previous section if $\Re(f_0(z))$ has a positive minimum over the unit torus, then the sufficient condition is indeed satisfied for sufficiently large values of T. If this function has minimum zero (recall that non-negativity of this function is a necessary condition) and $\Re(f_1(z))$ has a positive minimum, then again the sufficient condition is satisfied for large enough T. In fact more generally if there exists a value $0 < \lambda < 1$ such that $\Re(f_0(z) + \lambda f_1(z))$ has positive minimum then there exists a value of T such that the sufficient condition is satisfied (use $b_0(t) > b_1(t)/\lambda$ for sufficiently large T and replace b_0 by b_1/λ; as $\Re(f_0) \geq 0$ the result follows). Of course for any given EPT function one could try to analyze whether it satisfies the sufficient conditions using these or other "ad hoc" arguments. In what follows we will present a more systematic method to analyze the problem. However in general it will require some additional arguments to be able to conclude that the sufficient condition is satisfied with mathematical certainty.

We first introduce some notation. Let $\Phi(z) := (\Phi_1(z), \ldots, \Phi_N(z))'$ with $\Phi_k(z_1, z_2, \ldots, z_m) = \frac{1}{2} f_k(z_1, z_2, \ldots, z_m) + \frac{1}{2} \bar{f}_k(z_1^{-1}, z_2^{-1}, \ldots, z_m^{-1}),\ k = 0, 1, 2, \ldots, N$, mapping the unit torus $\mathbf{T}^m$ into $\mathbf{R}^{N+1}$. Let $F = \Phi(\mathbf{T}^m)$ denote the subset of $\mathbf{R}^{N+1}$ of points in the image of Φ.

Let $b(t) = (b_0(t), b_1(t), \ldots, b_N(t))$ and let $T > 0$ be given. Define the following subsets of $\mathbf{R}^{N+1}$ that relate to non-negativity and positivity of linear combinations of the EP functions $b_0(t), b_1(t), \ldots, b_N(t))$ for $t \geq T$:

$$Q := \{\phi \in \mathbf{R}^{N+1} | b(t)\phi \geq 0, \forall t \in [T, \infty)\},$$

$$P := \{\phi \in \mathbf{R}^{N+1} | b(t)\phi > 0, \forall t \in [T, \infty)\}.$$

Let

$$Z := Q \backslash P,$$

and

$$\Delta := \partial Q,$$

the boundary of Q in $\mathbf{R}^{N+1}$.

Remark Note that $Z \subseteq \Delta,\ Z \neq \Delta$. Consider for instance

$$e^{-\alpha_1 t}\phi_1 + e^{-\alpha_2 t}\phi_2,\ 0 < \alpha_1 < \alpha_2,$$

Then $(\phi_1, \phi_2) = (0, 1) \in \Delta \backslash Z$, as $(-\epsilon, 1) \notin Q, \forall \epsilon > 0; (0, 1) \in Q$. □

Define a projection

$$\mathbf{R}^{N+1} \to \mathbf{R}^{N+1}, \phi = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} \mapsto R(\phi) = \begin{pmatrix} \rho_0 \\ \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$$

by $\rho_0 = \text{Min}\{\tilde{\rho}_0 | \begin{pmatrix} \tilde{\rho}_0 \\ \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} \in Q\}$

Note that $\rho_0 \geq 0$ (due to the dominance of $b_0(t)$ over $b_1(t), \ldots, b_N(t)$).

Let $L(\phi) := \phi_0 - \rho_0 = e_1'(\phi - R(\phi)),\ \forall \phi \in F$, where $e_1, e_2, \ldots, e_{N+1}$ form the standard basis for $\mathbf{R}^{N+1}$.

This function has some interesting properties.

Proposition 4.1. *(i) The function L is Lipschitz continuous with respect to the L_1 norm on $\mathbf{R}^{N+1}$, and $|L(\phi) - L(\tilde{\phi})| \leq |\phi - \tilde{\phi}|_{L^1}$.*

(ii) $L(\phi) \geq 0 \Longleftrightarrow \phi \in Q$.

Proof. ad(i) This can be shown using the fact that

$$e_1 - e_j = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in Q, \ \forall j \in \{2, 3, \ldots, N+1\},$$

which implies:

$$\tilde{\rho}_0 - \rho_0 \leq |\phi_2 - \tilde{\phi}_2| + |\phi_3 - \tilde{\phi}_3| + \ldots + |\phi_N - \tilde{\phi}_N|.$$

ad(ii) For any $\phi \in Q$ increasing the first entry gives another vector in Q, because it corresponds to adding a positive multiple of b_0 to a non-negative EP function, which gives another non-negative EP function. It follows, by construction of L, that if $L(\phi) \geq 0$ then $\phi \in Q$. Also if $\phi \in Q$ then by construction $L(\phi) \geq 0$, hence the statement in the proposition follows. □

This result can now be applied as follows.

Let T be fixed. We can parametrize the torus $\mathbf{T}^m$ by a map $Z : (\mathbf{R}/\mathbf{Z})^m \to \mathbf{T}^m$ that is given by $\omega = (\omega_1, \omega_2, \ldots, \omega_m) \mapsto Z(\omega) = (e^{2\pi i \omega_1}, e^{2\pi i \omega_2}, \ldots, e^{2\pi i \omega_m})$. Here the ω_j are taken to be real numbers between zero and one. Composing this with Φ one can parametrize F by $F = \Phi(Z([0,1)^m))) = \Phi(Z([0,1]^m)))$. Now note that $\mathrm{Min}_{\phi \in F} L(\phi)$ is non-negative if and only if $F \subset Q$ if and only if the sufficient condition holds. As F is compact ($\mathbf{T}^m$ is compact and L is continuous), this minimum exists. We can build a grid in $[0,1]^m$ with a prescribed mesh size $\mu > 0$, i.e. every point in the set is at a distance of at most μ to a point in the grid. Because the composition $\Phi(Z(\omega))$ is real analytic and L is Lipschitz continuous the composition $L(\Phi(Z(\omega)))$ is Lipschitz continuous as well and a uniform Lipschitz constant can be found for this mapping using the previous proposition and the fact that the derivatives of $\Phi(Z(\omega))$ consist of trigonometric polynomials each of which can be bounded by the sum of the absolute values of their coefficients. Therefore one can construct a lower bound for the minimum within distance $\epsilon > 0$ of the minimum, for arbitrarily small positive ϵ by calculating $L(\Phi(Z(\omega)))$ on the grid with sufficiently small mesh size μ. That leaves the question on how $L(\phi)$ can actually be calculated for a given vector $\phi = \Phi(Z(\omega))$. Here we can use the fact that for any EP function $y(t) = b(t)\phi$ non-negativity can be verified by determining a T such that $y(t) > 0$ for all $t > T$ and determining the minimum of $y(t)$ on $[0, T]$ using the BF sequence algorithm. So we can check for each point in $\mathbf{R}^{N+1}$ whether it is in Q or not. The idea is now to apply the bisection technique to determine $R(\phi)$ and hence $L(\phi)$. Recall that $R(\phi) \geq 0$. In the bisection algorithm we will create vectors which are obtained from ϕ by replacing the first entry by another number, say $r = r(n)$, where $n = 0, 1, 2, \ldots$ denote the stages in the algorithm. Note that $r \geq 0$ must hold, because $r < 0$ gives a vector outside Q. If taking $r = 0$ gives a vector in Q then $R(\phi) = 0$ and we can stop. Now assume taking $r = 0$ produces a vector outside Q and let $r(0) := 0$. If $\phi \in Q$ we can take $r(1) = \phi_0$; otherwise one can take $r(1) = |\phi_1| + |\phi_2| + \ldots + |\phi_N|$ to be sure that we obtain a vector in Q (this follows from the fact that $b_0 \geq b_1 \geq \ldots \geq b_N$). Now do bisection to create a sequence $r(n),\ n = 0, 1, 2, \ldots$ where each next value in the sequence is a mid-point between two earlier values in the sequence depending on whether the various corresponding vectors are inside or outside Q. At each stage the pair $r(n)$ and $r(n+1)$ corresponds to a pair of vectors one of which is inside Q and the other one is outside Q. This sequence will converge and the limit will lie in Q, because Q is closed! So this gives us $R(\phi)$ and hence also $L(\phi)$.

Note that if $\Re f_0(z)$ has minimum equal to zero and $\Re f_0(\hat{z}) = 0$ then $L(\Phi(\hat{z})) = R(\Phi(\hat{z})) = 0$. Therefore in that case the minimum of L on F is at most zero. Therefore this method will not give mathematical certainty about whether the sufficient condition is satisfied. On the other hand, if the sufficient condition is NOT satisfied, then for a sufficiently refined mesh the grid calculations should reveal this.

5. Conclusions and further research

Further research is needed in order to obtain systematic methods that can tell us with mathematical certainty whether a given EP function satisfies the sufficient condition presented here. It may be possible to obtain this using algebraic methods (exploiting the ring structure of the class of EP functions).

Also it would be interesting to study the properties of the class of EPT functions that satisfy the sufficient conditions, such as the possible invariance under certain operations (addition, multiplication etc). Gaining a better understanding in the class of EPT functions that are non-negative but do NOT satisfy the sufficient conditions is also of interest; it may lead to number theoretical considerations.

To verify the necessary condition one needs to verify whether a certain multivariable trigonometric polynomial is non-negative. Non-negativity of polynomials including trigonometric polynomials is a topic of active research at the moment, where techniques from constructive algebra, real algebraic geometry and numerical optimization (such as interior point methods for convex optimization) are combined.

For practical applications it will be important to see how one can deal with questions of linear dependence and independence of certain numbers over the rational numbers. Whether the practical problem formulation is specified sufficiently carefully so that such questions can be answered remains to be seen. However if the EPT functions are obtained by operating on other EPT functions (addition, multiplication, convolution etc.,) rational dependencies may well show up explicitly.

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