MATLAB CODE FOR EPT-CALCULATIONS

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November 2011

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1. Introduction

There are two distinct segments of code available for download. The first segment of code is for the Budan-Fourier algorithm of Hanzon and Holland (2010) to locate all sign-changing zeros of an EPT function on a finite interval $[0, T]$. This programme is compiled of six m-files. The second function available computes the additive decomposition of a rational function using a single m-file. The reader is referred to Hanzon and Sexton (2011) for a detailed description of EPT functions. An example of Budan-Fourier algorithm is provided here to assist in understanding the code.

All versions of code uploaded are still only in testing stages and may not be free of bugs. Users reporting errors or queries about the files should contact the authors at `b.hanzon@ucc.ie` or `hcsexton@hotmail.com`.
The additive decomposition code was originally written by Prof. Wolfgang Scherrer and we are extremely grateful for his permission to make the file available. However all queries should directed towards the authors.
The code should be copied and saved as individual .m files, according to the name of the function, in the same directory.

2. Budan-Fourier

2.1. Zeros_EPT.m

This is the primary m-file for the Budan-Fourier algorithm. The output is a plot of the EPT function identifying any sign-changing zeros on the interval $[0, T]$ and also displaying the location of these points. If there are no sign-changing zeros on the interval then a statement indicating this is printed. There are four inputs to the function which includes the $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ minimal realization defining the EPT function. $\mathbf{A}$ is an $(n \times n)$ square matrix, $\mathbf{b}$ an $(n \times 1)$ column vector and $\mathbf{c}$ a $(1 \times n)$ row vector. "T" indicates the upper bound of the interval $[0, T]$ on which to check for sign-changing zeros. The plotting frequency, N, of the output is set at 500 by default but can be changed easily. The structure of the spectrum of $\mathbf{A}$ is then examined. Depending on the eigenvalues, a different sub-routine is called to determine the zeros of the EPT function. If all eigenvalues are zero implying the function is a polynomial then `Budan_Fourier_P.m` is called while if the eigenvalues of $\mathbf{A}$ are real and non-zero `Budan_Fourier_EP.m` is utilised indicating an exponential-polynomial function is being examined. If complex eigenvalues are present `Budan_Fourier_EPT.m` is called implying an exponential-polynomial-trigonometric function. Each of the Budan Fourier algorithms return a vector of zeros (excluding the end-points, "0" & "T") which are potentially sign-changing zeros. We choose a small epsilon, "eps", to examine whether each "x" is truly sign-changing. "x" is deemed to be sign-changing if and only if

$$sign(f(x - eps)) \times sign(f(x + eps)) = -1$$

If this condition is satisfied for a given “x” then it will appear in the output as a sign-changing zero.

2.2. Budan_Fourier_P.m

If all eigenvalues of **A** are zero such that the EPT function is a polynomial then `Budan_Fourier_P.m` is called which returns the zeros of the EPT function. The zeros are calculated by computing the simple grid from the Budan-Fourier sequence (derivatives in this case) of the EPT function. The zeros on the simple grid are located using a bisection algorithm in `EP_Bisection`.

2.3. Budan_Fourier_EP.m

This m-file is called if all eigenvalues of **A** are real and not all zero. Again the zeros of the EPT function are returned as a vector. The Budan-Fourier sequence for exponential-polynomials is found using the Cayley-Hamilton result as given in Hanzon and Holland (2010). The zeros on the simple grid are located using the `EP_Bisection.m` algorithm.

2.4. Budan_Fourier_EPT.m

The zeros of an EPT function which has complex spectrum are returned. A generalised Budan-Fourier sequence with boundary points is formed as described in Hanzon and Holland (2010). Zeros can be located on a simple grid using the `EPT_Bisection` method.

2.5. EP_Bisection.m

The inputs are the triple of the EP function which has a real spectrum and also the lower and upper boundaries, “l” and “r” resp., of the simple interval. The algorithm assumes there is one zero on the interval $[l, r]$ and locates that zero up a tolerance of 10^{-6}. The zero on the grid is returned. The tolerance, “tol”, can be changed easily.

2.6. EPT_Bisection.m

The inputs are the triple of the EPT function which has a complex spectrum and also the lower and upper boundaries, “l” and “r” resp., of the simple interval. The eigenvalue λ_k associated with the latest sequence of the Budan-Fourier algorithm is also passed as an input. A final additional input is ”S” which is an indicator function, 0 if the eigenvalue λ_k is real and $S = 1$ if λ_k is complex. The algorithm assumes there is only one zero on the interval $[l, r]$ and locates that zero up a tolerance of 10^{-6}. The zero on the grid is returned. This bisection process is more involved than `EP_Bisection.m` as it depends on the whether λ_k is real or complex.

3. AdditiveDecomposition.m

The purpose of the file is to decompose a rational function $\phi(u)$ given by the realization $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ of order n, with $n/2$ poles in either half plane, into the sum of two rational functions $\phi_P(u)$ and $\phi_N(u)$.

$$\phi(u) \; = \; \phi_P(u) + \phi_N(u)$$

$\phi_N(u)$ is a rational function with realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N)$ whose poles are located in the open right plane. $\phi_P(u)$ is another rational function with poles located in the open left half plane with realization $(\mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$. The `cschur` function within the script computes a unitary similarity transformation which is used as a basis change. The resultant "A" matrix is upper triangular with eigenvalues (i.e. diagonal elements) sorted in ascending order according to their real part. The matrix is then transformed to block diagonal form by solving the appropriate sylvester equation using `lyap` function. The basis change is performed on the complete triple $(\mathbf{A}, \mathbf{b}, \mathbf{c})$. The inputs required are the minimal 4-tuple $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ and the output is two minimal triples $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N)$ where the spectrum of $\mathbf{A}$ located in the open right half plane and $(\mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$ which has its spectrum on the open left half plane. Although $\mathbf{d}$ can given as an input it plays no part in the additive decomposition and is returned unchanged.

4. Example I

```
v=[0;i;-i;i*pi;-i*pi;-1];
A=diag(v);
b = [2;1/2;1/2;1/2;1/2;-20];
c = [1 1 1 1 1 1 ];
Zeros_EPT(A,b,c,10)
```

The code should plot the EPT function,

$$\mathbf{c}e^{\mathbf{A}x}\mathbf{b} \;=\; 2+\frac{1}{2}e^{ix}+\frac{1}{2}e^{-ix}+\frac{1}{2}e^{i\pi x}+\frac{1}{2}e^{-i\pi x}-20e^{-x}$$

on $[0,10]$ and identify three sign-changing zeros at $x \;=\; \{2.1049, 2.2092, 3.3854\}$

5. Example II

Consider the Nelson-Siegel forward rate curve given by

$$f(t) \;=\; 2+3e^{-\frac{t}{4}}-\frac{1}{2}te^{-\frac{t}{4}}$$

The Laplace transform (transfer function) of $f(t)$ is the strictly proper rational function given by

$$\phi_f(u) \;=\; \frac{80u^2-4u+2}{16u^3+8u^2+u}$$

We can input the transfer function as a state space system using the `tf` and `ss` commands which takes as inputs the coefficients of the denominator and numerator. The commands `tf` and `ss` require the Control System Toolbox.

```
num = [80 -4 2]
den = [16 8 1 0]
Sys = tf(num, den)
Sys = ss(Sys)
Zeros_EPT(Sys.a, Sys.b, Sys.c, 10)
```

Two sign-changing zeros are identified at $t \;=\; (4.9286, 6.1317)$

References

Hanzon, B., Sexton, C., *State Space Calculations for two-sided EPT Densities with Financial Modelling Applications.* Forthcoming. Draft version available at “www.edgeworth.biz”, 2011

Hanzon, B., Holland, F. *Non-Negativity of Exponential Polynomial Trigonometric Functions - A Budan Fourier Sequence Approach.* BFS Toronto Poster 429, June 22 - 26, 2010. Online version available at “www.edgeworth.biz”