Financial Modelling with 2-EPT Levy Processes

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Abstract

The class of probability density functions on $\mathbb{R}$ with strictly proper rational characteristic functions are considered. On $[0,\ \infty)$ as well as $(-\infty,\ 0)$ these probability density functions are Exponential-Polynomial-Trigonometric (EPT) functions which we abbreviate with as 2-EPT densities. EPT density functions can be represented as $f(x) = ce^{Ax}b$, where "A" is a square matrix, "b" a column vector and "c" a row vector. The triple (A, b, c) is called the realization of the EPT density function. The more general class of probability measures on $\mathbb{R}$ with (proper) rational characteristic functions is also considered whose densities correspond to mixtures of the pointmass at zero ("delta distribution") and 2-EPT densities. Using results from Widder (1941) and Feller (1971) it is shown how to characterize infinitely divisible EPT functions. The Lévy triple of the infinitely divisible EPT can be derived it is seen that EPT Lévy processes are of finite variation. Similar results are proven for EPT functions mixed with a pointmass at zero. The Laplace transform of an infinitely divisible 2-EPT probability density function can be factored into two rational functions. These rational functions are the Laplace Transforms of EPT functions defined on either half line. The 2-EPT probability density function is infinitely divisible if and only if the EPT functions on the each halfline are infinitely divisible.

KeyWords : Rational Characteristic Function, Levy Process, Levy-Khintchine Formula, Infinite Divisibility

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1. Introduction

Much of the success of the Black-Scholes model can be attributed to the closed form risk neutral option prices it generates. However, such enviable results and analytic tractability stem from its simplistic modelling assumptions of a pure diffusion process in which log-returns have a Gaussian distribution. The assumption of continuous asset paths creates a complete market where delta hedging strategies are used to derive risk neutral option prices. Within this complete markets framework vanilla options are clearly redundant as they can be replicated with a portfolio of risky assets and bonds. However, once jumps are introduced to the price process, the continuous hedging arguments break down but lead to a more realistic, incomplete market. Therefore without dynamic hedging these options are no longer futile and become completing assets, hedging against jump risk. It has been well documented in the literature that the Black-Scholes assumptions of Gaussian returns and continuous asset paths cannot be justified by empirical studies.

Significant research has since been carried out seeking more realistic models which better capture the stylized characteristics of asset returns including excess kurtosis, skewness and jumps. The general class of Lévy processes have emerged as a viable alternative to Brownian Motion models due to their flexibility in this regard. Modelling with Lévy processes preserves the stationarity and independence of increments while allowing for jumps, distributional asymmetry and heavier tails than the Gaussian distribution. We will see that Lévy processes are characterized by their unique Lévy triple comprising of a drift component, a diffusion component and a Lévy measure. A Lévy measure $\nu(A)$ is the expected number, per unit time, of jumps whose size belong to A. Brownian Motion is a special case of a Lévy process with no jumps achieved by an identically zero Lévy measure. The relationship between Lévy processes and infinitely divisible distributions is very close as there exists an infinitely divisible distribution for every Lévy process. The converse is well known, and has been shown in Kyprianou and Loeffen (2005), that for every infinitely divisible distribution there exists a unique Lévy process.

For the purposes of derivatives pricing it is clear that the price process model must be free of arbitrage. The absence of arbitrage is equivalent to the existence of an equivalent martingale measure. In Delbaen and Schachermayer (1994) it is shown that if there exists a change of measure from $\mathbb{P}$ to the risk neutral measure $\mathbb{Q}$ such that under $\mathbb{Q}$ the discounted price process is a martingale then under $\mathbb{P}$ the price process must have been a semi-martingale. It is well known and can be seen in Cont and Tankov (2003) that all Lévy processes are semi-martingales. It is also known that any twice differentiable function of a semi-martingale is again a semi-martingale. Therefore all exponential Lévy processes are semi-martingales. Proposition 9.9 from Cont and Tankov (2003) proves that there exists an equivalent martingale measure $\mathbb{Q}$ if the underlying Lévy process has both positive and negative jumps which is the case for 2-EPT processes.

A downside associated with Lévy processes is that the infinitely divisible distributions generating such processes on $\mathbb{R}$ often have a complicated structure. A typical example of this problem is the generalised hyperbolic distribution of which the Variance Gamma and Normal-Inverse Gaussian are special cases. Both distributions are infinitely divisible but the specification of the density involves special functions making even simple calculations involving the density difficult. The Meixner distribution gives rise to Lévy processes popular in Mathematical Finance but the structure of its density has not yet lead to closed form option pricing formulae. For this reason, calculations involving these density functions often require numerical techniques.

The class of 2-EPT densities was introduced in Hanzon and Sexton (2012) and correspond to those probability density functions on $\mathbb{R}$ with a strictly proper characteristic function. On $[0,\ \infty)$ as well as $(-\infty,\ 0)$ these probability density functions are Exponential-Polynomial-Trigonometric (EPT) functions. EPT functions can be represented as $\mathbf{c}e^{\mathbf{A}x}\mathbf{b}$, where $\mathbf{A}$ is a square matrix, $\mathbf{b}$ a column vector and $\mathbf{c}$ a row vector. The triple $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ can be taken to be a minimal realization of an EPT function. We also consider generalised 2-EPT probability density functions which correspond to 2-EPT functions mixed with a pointmass at zero. The characteristic functions of the generalised 2-EPT probability density functions are proper rational functions. It is shown in Hanzon and Sexton (2012) that the class of densities is closed under many operations and that the Variance Gamma distribution is 2-EPT under the parameter restriction that the input parameter "C" is an integer. The Variance Gamma distribution is infinitely divisible and hence has an associated Lévy process, namely the Variance Gamma process. The Variance Gamma process is a pure jump Lévy process of finite variation which Madan (1999) has advocated for financial modelling purposes. Assuming that log-returns for an asset are modelled using a Variance Gamma process then closed form option pricing formulae can be derived when $C\tau$ is an integer, where τ is the time to maturity of the option. Analytic formulae also exists for the greeks of these option prices.

It will seen that infinitely divisible 2-EPT probability density functions generate pure jump Lévy processes with no diffusion component known as 2-EPT Léy processes. These 2-EPT Lévy processes can be used to model log-returns of an asset and over a fixed period τ the log returns have a 2-EPT probability density function. Under these assumptions close form option pricing formulae for options, with time τ to maturity, can then be derived as in Hanzon and Sexton (2012). The Greeks of such options can also be computed in closed form.

2. Mathematical Formulation

We begin with the definition of a Levy process from Cont and Tankov (2003)

Lévy Process *A càdlàg continuous from stochastic process (Paths are right-continuous with left limits everywhere, with probability one)* $(X_t)_{t\geq 0}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with values in* $\mathbb{R}$ *such that* $X_0 = 0$ *is called a Lévy process if it possesses the following properties*

- *Independent increments: for every increasing sequence of times* $t_0, t_1, ..., t_n$*, the random variables* $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ *are independent.*
- *Stationary increments: the law of* $X_{t+h} - X_t$ *does not depend on* t*.*
- *Stochastic continuity:* $\forall\ \epsilon > 0,\ \lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \epsilon) = 0.$

The Lévy-Khintchine formula is used to characterise infinitely divisible distributions in terms of its Lévy triple and from Kyprianou and Loeffen (2005)

Lévy-Khintchine Formula *A probability law* f *of a real valued random variable is infinitely divisible with characteristic exponent* Ψ*,*

$$\int_{\mathbb{R}} e^{-isx} f(x)dx \ = \ e^{-\Psi(s)}, \qquad s \ \in \ \mathbb{R}, \tag{1}$$

if and only if there exists a triple (a, σ, ν) *where* $a \in \mathbb{R}$*,* $\sigma \geq 0$ *and* ν *a measure concentrated on* $\mathbb{R}\backslash\{0\}$ *satisfying* $\int_{\mathbb{R}} \min\{1, x^2\} d\nu(x) \ < \ \infty$*, such that*

$$\Psi(s) \ = \ ias + \frac{1}{2}\sigma^2 s^2 + \int_{\mathbb{R}} (1 - e^{isx} + isx\mathbb{I}_{\{|x|<1\}})d\nu(x), \tag{2}$$

for all s *in* $\mathbb{R}$*.*

The measure ν is the unique Lévy measure. The characteristic exponent from Eq. (2) can be decomposed as

$$\Psi(s) \ = \ ias + \frac{1}{2}\sigma^2 s^2 + \int_0^{\infty} (1 - e^{isx} + isx\mathbb{I}_{\{x<1\}})d\nu_P(x) + \int_{-\infty}^{0} (1 - e^{isx} + isx\mathbb{I}_{\{x>-1\}})d\nu_N(x), \tag{3}$$

where $\nu_P(x) = 0$ for all $x < 0$ and $\nu_N(x) = 0$ for all $x > 0$.

Proposition 3.9 from Cont and Tankov (2003) provides a formulation for Lévy processes of finite variation.

Finite Variation Lévy Processes *A Lévy process is of finite variation if and only if its characteristic triplet* (a, σ, ν) *satisfies*

$$\sigma = 0 \qquad \textit{and} \qquad \int_{|x|\leq 1} |x| d\nu(x) \ < \ \infty \tag{4}$$

The open right half plane is defined as $\mathbb{H}_+ \ = \ \{\omega \ : \ Re(\omega) > 0\}$ and the corresponding open left half plane is defined as $\mathbb{H}_- \ = \ \{\omega \ : \ Re(\omega) < 0\}$.

3. Infinitely Divisible EPT Distributions

We begin by examining an EPT probability density function f defined on $[0,\infty)$ by the minimal realization $(\mathbf{A},\mathbf{b},\mathbf{c})$

$$f(x) \;=\; \mathbf{c}e^{\mathbf{A}x}\mathbf{b}, \qquad x \geq 0, \tag{5}$$

such that $Re(\sigma(\mathbf{A})) \subset \mathbb{H}_-$ where $\sigma(\mathbf{A})$ denotes the spectrum of $\mathbf{A}$. As f is non-negative on the half line $[0,\infty)$ it can be seen in Hanzon and Holland (2010b) that a Perron-Frobenius type result implies that the spectrum of $\mathbf{A}$ contains a negative dominant real eigenvalue λ_M such that $\lambda_M = \max_{\lambda\in\sigma(A)} Re(\lambda)$.
The Laplace transform of f for $s > 0$ is given by

$$F(s) \;=\; \int_0^\infty e^{-sx}f(x)dx \;=\; \int_0^\infty e^{-sx}\mathbf{c}e^{\mathbf{A}x}\mathbf{b}dx \;=\; \mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b} \;=\; \frac{p(s)}{q(s)}. \tag{6}$$

F is a strictly proper rational function as p and q are co-prime polynomials of order m and n respectively where $n > m$.
We let $\Lambda(\mathbf{A},\mathbf{b},\mathbf{c}) = \{s|\mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b} = 0\}$ be the set of zeros of the function $\mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}$. A result from Lukacs (1970) states that an infinitely divisible analytic function cannot have any zero inside its strip of convergence. If a rational function $\mathbf{c}(s\mathbf{I}-A)^{-1}\mathbf{b}$ is infinitely divisible it must hold that $\max_{\lambda\in\Lambda(\mathbf{A},\mathbf{b},\mathbf{c})} Re(\lambda) \leq \lambda_M$. Only triples $(\mathbf{A},\mathbf{b},\mathbf{c})$ satisfying this necessary condition will be considered, otherwise the resulting EPT probability density function is not infinitely divisible. As $\lambda_M < 0$ it holds that $\Lambda(\mathbf{A},\mathbf{b},\mathbf{c}) \subset \mathbb{H}_-$. In systems theory a transfer function with all poles and zeros located in the half plane is referred to as "minimum phase".
It is known, Feller (1971), that a function g is the Laplace transform of an infinitely divisible probability distribution on $[0,\infty)$, if and only if $g = e^{-w}$ where the derivative of w is completely monotonic on $[0,\infty)$ and $w(0+) = 0$.
Completely Monotonic Function *A function J defined on $(0,\infty)$ is said to be completely monotonic (c.m.) if it possesses derivatives $J^n(x)$ for all $n = 0,1,2,...$ such that*

$$(-1)^n J^{(n)}(x) \;\geq\; 0, \tag{7}$$

for all $x > 0$
It is known from Bernstein Theorem, given in Widder (1941), that a necessary and sufficient condition that a function J is c.m. on $[0,\infty)$ for $0 \leq s < \infty$ if

$$J(s) \;=\; \int_0^\infty e^{-sx}d\alpha(x), \tag{8}$$

where $\alpha(x)$ is a non-negative measure on $[0,\infty)$ and the integral converges for $0 < s < \infty$. We conclude that a non-identically zero c.m. function $J(s)$ cannot vanish for any positive s.
From the results of Feller (1971) above, if F is the Laplace transform of the infinitely divisible distribution f on $[0,\infty)$ then $F = e^{-h}$ and the derivative of h given by

$$h' \;=\; -\frac{F'}{F}, \tag{9}$$

must be c.m. on $[0, \infty)$. By construction it holds that $h(0) = -\log(F(0)) = 0$ as $F(0) = 1$ since f is a probability density function. It is clear that if h' is completely monotonic on $[0, \infty)$ then there exists some positive measure ν that is not necessarily finite such that

$$h'(s) \;=\; \int_0^\infty e^{-sx} d\nu(x), \qquad s > 0. \tag{10}$$

Since $F = e^{-h}$ and $h(0) = 0$, by Fubini

$$h(s) \;=\; \int_0^s h'(t)dt \;=\; \int_0^\infty \frac{1-e^{-sx}}{x} d\nu(x), \qquad s > 0. \tag{11}$$

Lemma 1 *Let $F = e^{-h}$, where F is a minimum phase Laplace Transform of a probability density function. Then h' is completely monotonic if there exists a positive measure ν such that*

$$h(s) \;=\; \int_0^\infty \frac{1-e^{-sx}}{x} d\nu(x), \tag{12}$$

exists and is analytic for all $s \in \mathbb{H}_+$

Proof

We have that

$$\int_0^\infty \frac{1-e^{-x}}{x} d\nu(x) \;=\; h(1) \;<\; \infty.$$

However

$$\frac{1-e^{-x}}{x} \;>\; \frac{1}{1+x}, \qquad x \;>\; 0,$$

and so

$$\int_0^\infty \frac{1}{x+1} d\nu(x) \;<\; \infty.$$

It follows from this that

$$\int_1^\infty \left|\frac{1-e^{-sx}}{x}\right| d\nu(x) \;\le\; 2\int_1^\infty \frac{1}{x} d\nu(x) \;<\; \infty, \qquad s \in \mathbb{H}_+,$$

whence

$$\int_0^\infty \frac{1-e^{-sx}}{x} d\nu(x) \;=\; \int_0^1 \frac{1-e^{-sx}}{x} d\nu(x) + \int_1^\infty \frac{1-e^{-sx}}{x} d\nu(x),$$

exists and is analytic for all $s > 0$ proving Lemma 1. ■

A limiting argument assures us that

$$h(i\omega) \;=\; \int_0^\infty \frac{1-e^{-i\omega x}}{x} d\nu(x), \qquad -\infty < \omega < \infty\,.$$

$\nu(x)$ defined on $(0, \infty)$ must be determined such that $h'(s)$ is the Laplace transform of $\nu(x)$. It is clear that

$$h'(s) \;=\; -\frac{F'(s)}{F(s)} \;=\; \frac{\mathbf{c}(s\mathbf{I}-\mathbf{A})^{-2}\mathbf{b}}{\mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}}.$$

The strictly proper rational function F can be written as

$$F(s) \;=\; \frac{\mathbf{c}(s\mathbf{I}-\mathbf{A})^{*}\mathbf{b}}{\det(s\mathbf{I}-\mathbf{A})}, \tag{13}$$

where the zeros of the polynomial $\mathbf{c}(s\mathbf{I}-\mathbf{A})^*\mathbf{b}$ are the zeros of $F(s)$. $\Big[\mathbf{M}^*$ denotes the adjoint matrix of the square matrix $\mathbf{M}$, such that $\mathbf{M}^*\mathbf{M} = \mathbf{M}\mathbf{M}^* = \det(\mathbf{M})\,\mathbf{I}\Big]$.

Lemma 2 *Suppose q is the characteristic polynomial of a square matrix $\boldsymbol{Q}$ where $\sigma(\boldsymbol{A}) \subset \mathbb{H}_-$. Then, for $s > 0$, $q'(s)/q(s)$ is the Laplace transform of the trace of* $\exp(\boldsymbol{Q}x)$

Proof

$$Tr(e^{\mathbf{Q}x}) \;=\; \sum_{n=0}^{\infty} \frac{x^n Tr(\mathbf{Q}^n)}{n!},$$

but

$$Tr(\mathbf{Q}^n) \;=\; \sum_{\lambda\in\sigma(\mathbf{Q})} \lambda^n.$$

Hence for $s > 0$

$$\begin{aligned}\int_0^\infty e^{-sx} Tr(e^{\mathbf{Q}x})dx &= \sum_{\lambda\in\sigma(\mathbf{Q})} \int_0^\infty \Big(\sum_{n=0}^{\infty} \frac{x^n\lambda^n}{n!}\Big) e^{-sx}dx \\ &= \sum_{\lambda\in\sigma(\mathbf{Q})} \int_0^\infty e^{\lambda x} e^{-sx} dx \\ &= \sum_{\lambda\in\sigma(\mathbf{Q})} \frac{1}{s-\lambda} \\ &= \frac{q'(s)}{q(s)}. \qquad \blacksquare\end{aligned}$$

Lemma 3 *Let F be a minimum phase strictly proper rational function and $F = e^{-h}$. Then $h' = -F'/F$, is the Laplace transform of $Tr(e^{\boldsymbol{A}x}) - Tr(e^{\boldsymbol{B}x})$ where $\boldsymbol{B}$ is the companion matrix of $\boldsymbol{c}(s\boldsymbol{I}-\boldsymbol{A})^*\boldsymbol{b}$ if* $\deg(\boldsymbol{c}(s\boldsymbol{I}-\boldsymbol{A})^*\boldsymbol{b}) > 0$. *If $\boldsymbol{c}(s\boldsymbol{I}-\boldsymbol{A})^*\boldsymbol{b}$ is constant then h' is the Laplace Transform of $Tr(e^{\boldsymbol{A}x})$.*

Proof

If $\deg(\mathbf{c}(s\mathbf{I}-\mathbf{A})^*\mathbf{b}) > 0$ let $F = p/q$ where $p(s) = \mathbf{c}(s\mathbf{I}-\mathbf{A})^*\mathbf{b}$ and $q(s) = \det(s\mathbf{I}-\mathbf{A})$. Using the quotient rule

$$-\frac{F'(s)}{F(s)} \;=\; \frac{q'(s)}{q(s)} - \frac{p'(s)}{p(s)}.$$

As F is of minimum phase, $Tr(e^{\mathbf{A}x})$ and $Tr(e^{\mathbf{B}x})$ are integrable on $(0,\infty)$ and by Lemma 2

$$\begin{aligned}-\frac{F'(s)}{F(s)} &= \int_0^\infty e^{-sx} Tr(e^{\mathbf{A}x})dx \;-\; \int_0^\infty e^{-sx} Tr(e^{\mathbf{B}x})dx \\ &= \int_0^\infty e^{-sx}\Big(Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}x})\Big)dx.\end{aligned}$$

If $\mathbf{c}(s\mathbf{I}-\mathbf{A})^*\mathbf{b} = K$ where $K \in \mathbb{R}$ then $F(s) = K/q(s)$ and $q(s) = \det(s\mathbf{I}-\mathbf{A})$. It can be seen that

$$-\frac{F'(s)}{F(s)} \;=\; \frac{q'(s)}{q(s)}, \tag{14}$$

and similarly

$$-\frac{F'(s)}{F(s)} \;=\; \int_0^\infty e^{-sx} Tr(e^{\mathbf{A}x})dx. \qquad \blacksquare \tag{15}$$

Theorem 1 *Given a minimal triple $(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c})$ that defines a not identically zero EPT probability density function whose Laplace Transform is of minimum phase. If $\deg(\boldsymbol{c}(s\boldsymbol{I}-\boldsymbol{A})^{*}\boldsymbol{b}) > 0$, $\boldsymbol{B}$ is the companion matrix of $\boldsymbol{c}(s\boldsymbol{I}-\boldsymbol{A})^{*}\boldsymbol{b}$ and $F(s)$ is the Laplace transform of $f(x)=\boldsymbol{c}e^{\boldsymbol{A}x}\boldsymbol{b}$, then*

$$F(s) \ = \ \exp\Big(-\int_0^\infty (1-e^{-sx})\frac{Tr(e^{\boldsymbol{A}x})-Tr(e^{\boldsymbol{B}x})}{x}dx\Big), \tag{16}$$

Thus F is infinitely divisible iff $Tr(e^{\boldsymbol{A}x}) \geq Tr(e^{\boldsymbol{B}x})$ for all $x \geq 0$
If $\boldsymbol{c}(s\boldsymbol{I}-\boldsymbol{A})^{*}\boldsymbol{b}$ is constant then the Laplace transform of $f(x)=\boldsymbol{c}e^{\boldsymbol{A}x}\boldsymbol{b}$ is given by $F(s)$

$$F(s) \ = \ \exp\Big(-\int_0^\infty (1-e^{-sx})\frac{Tr(e^{\boldsymbol{A}x})}{x}dx\Big), \tag{17}$$

where $F(s)$ is infinitely divisible iff $Tr(e^{\boldsymbol{A}x}) \geq 0$ for all $x \geq 0$.
By comparing the exponent of $F(s)$ in Eq. (16) to the characteristic exponent of the piecewise constant Lévy process in Eq. (**??**) the Lévy triple (a,σ,ν) of the EPT probability density function with realization $(\mathbf{A},\mathbf{b},\mathbf{c})$ can be determined. The drift component a must be chosen such that

$$a \ = \ -\int_0^1 x \ d\nu(x) \tag{18}$$

while the diffusion term $\sigma = 0$ and if $\deg(\mathbf{c}(s\mathbf{I}-\mathbf{A})^{*}\mathbf{b}) > 0$ the Lévy measure is given by

$$\nu(x) \ = \ \frac{Tr(e^{\mathbf{A}x})-Tr(e^{\mathbf{B}x})}{x}, \qquad x > 0, \tag{19}$$

and $\nu(x)=0$ for all $x<0$ while if $\deg(\mathbf{c}(s\mathbf{I}-\mathbf{A})^{*}\mathbf{b}$ is constant then

$$\nu(x) \ = \ \frac{Tr(e^{\mathbf{A}x})}{x}, \qquad x > 0. \tag{20}$$

Again we must have $\nu(x)=0$ for all $x<0$.
According to the definition in Eq. (4) the Lévy process is of finite variation if a from Eq. (18) is finite. Consider the case where $\deg(\mathbf{c}(s\mathbf{I}-\mathbf{A})^{*}\mathbf{b} > 0$ and note that

$$\begin{aligned} a &= -\int_0^1 \Big(Tr(e^{\mathbf{A}x})-Tr(e^{\mathbf{B}x})\Big)dx \\ &= -\Big[Tr(\mathbf{A}^{-1}e^{\mathbf{A}x})-Tr(\mathbf{B}^{-1}e^{\mathbf{B}x})\Big]_0^1 \\ &= -Tr(\mathbf{A}^{-1}e^{\mathbf{A}})+Tr(\mathbf{B}^{-1}e^{\mathbf{B}})+Tr(\mathbf{A}^{-1})-Tr(\mathbf{B}^{-1}) \ < \ \infty \end{aligned} \tag{21}$$

which holds as $\{\sigma(\mathbf{A})\cup\sigma(\mathbf{B})\}\subset\mathbb{H}_{-}$. When $\mathbf{c}(s\mathbf{I}-\mathbf{A})^{*}\mathbf{b}$ is constant it can be seen that

$$a \ = \ -Tr(\mathbf{A}^{-1}e^{\mathbf{A}})+Tr(\mathbf{A}^{-1}) \tag{22}$$

Therefore Lévy processes generated from infinitely divisible EPT distributions are of finite variation.
Consider the case where $\deg(\mathbf{c}(s\mathbf{I}-\mathbf{A})^{*}\mathbf{b}) > 0$, then $x\nu(x)$ is an Exponential-Trigonometric (ET) function given by

$$x\nu(x) \ = \ Tr(e^{\mathbf{A}x})-Tr(e^{\mathbf{B}x}) \tag{23}$$

It is clear that F defined in Eq. (16) is infinitely divisible if and only if $x\nu(x)$ is non-negative on $[0,\infty)$.

It has been stated that $\sigma(\mathbf{A})$ contains a dominant real eigenvalue $\lambda_M < 0$. Denote $\tilde{\lambda}_M = \max_{\lambda\in\sigma(\mathbf{B})} Re(\lambda)$ as the eigenvalue pole in $\mathbf{B}$. It is clear that if $x\nu(x) \geq 0$ in Eq. (23) for all $x \geq 0$ then it must hold that $\lambda_M \geq \tilde{\lambda}_M$. If $\lambda_M < \tilde{\lambda}_M$ then there would exist a $K > 0$ such that

$$Tr(e^{\mathbf{A}K}) < Tr(e^{\mathbf{B}K}). \tag{24}$$

This coincides with the result from Lukacs (1970) such that an analytic characteristic function of an infinitely divisible distribution can have no zeros located in its strip of convergence. This property can be used as a quick check to confirm that a characteristic function $F(s)$ is not infinitely divisible.

If F as given in Theorem 1 is infinitely divisible then it is clear that

$$\begin{aligned}
\nu(x) = \quad & \frac{Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}x})}{x} \geq 0, && \forall x > 0 \\
\iff \quad & Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}x}) \geq 0, && \forall x > 0 \\
\implies \quad & \lim_{x\downarrow 0} Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}x}) \geq 0 \\
= \quad & Tr(\mathbf{I_A}) - Tr(\mathbf{I_B}) \geq 0
\end{aligned} \tag{25}$$

where $\mathbf{I_A}$ is an identity matrix with the same dimensions as the square matrix $\mathbf{A}$. It follows that $\dim(\mathbf{A}) \geq \dim(\mathbf{B})$ must hold if F is the Laplace Transform of an infinitely divisible EPT distribution implying that F must not contain more zeros than poles.

The Budan-Fourier Algorithm of Hanzon and Holland (2010a) can be used to test for non-negativity of an EPT function on a finite interval. If $\max_{\lambda\in\{\sigma(\mathbf{A})\setminus\lambda_M\}} Re(\lambda) < \lambda_M$ the aforementioned Budan-Fourier technique can be used to test for non-negativity of the EPT function on the whole half line $[0,\infty)$. This result is proven in Hanzon, Olivi & Sexton (2012).

A necessary condition that the function f is infinitely divisible is that f is strictly positive on $[0,\infty)$ and this can be seen in Lemma 4.

Lemma 4 *Suppose the minimal triple $(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c})$ defines a non-negative EPT function denoted by f, that is not identically zero, which in turn defines a finite measure $f(x)dx$. If f is infinitely divisible it is necessary that f is strictly positive on $[0,\infty)$.*

Proof

Consider the Laplace Transform of f given by

$$F(s) = \int_0^\infty e^{-sx} f(x)dx = \mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}, \tag{26}$$

for $s \in \mathbb{H}_+$. Then

$$-\frac{F'(s)}{F(s)} = \frac{\mathbf{c}(s\mathbf{I}-\mathbf{A})^{-2}\mathbf{b}}{\mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}}, \tag{27}$$

which is a rational function. Hence. $-F'/F$ is the Laplace Transform of some continuous function k such that for all $s \in \mathbb{H}_+$

$$-\frac{F'(s)}{F(s)} = \int_0^\infty e^{-sx}k(x)dx. \tag{28}$$

Similarly

$$\int_0^\infty e^{-sx} x f(x)dx = \int_0^\infty e^{-sx} f(x)dx \int_0^\infty e^{-sx}k(x)dx, \tag{29}$$

which is the convolution of f and h. By uniqueness of the transform,

$$xf(x) \;=\; \int_0^\infty f(x-t)h(t)dt,\; 0 < x < \infty. \tag{30}$$

By assumption $f \;\geq\; 0$. If F is infinitely divisible then as noted earlier $-F'/F$ is completely monotonic. Hence $-F'/F$ is the Laplace Transform of some not necessarily finite non-negative measure. Therefore $k \geq 0$.
Finally if $f(a) = 0$ for some $a > 0$, then

$$0 \;=\; af(a) \;=\; \int_0^a f(a-t)h(t)dt, \tag{31}$$

whence, by continuity, $f(a-t)h(t) \equiv 0,\; \forall\, t \;\in\; [0, a]$, which is impossible and the result follows.

4. Infinitely Divisible EPT Distributions Mixed with Dirac Function

We consider an EPT function mixed with a pointmass at zero such that the probability density function f defined on $[0, \infty)$ has a minimal realization $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ of McMillan degree n and $0 < \mathbf{d} < 1$.

$$f(x) = \begin{cases} 0 & \text{with probability } \mathbf{d} \\ \\ \mathbf{c}e^{\mathbf{A}x}\mathbf{b} & \text{if } x > 0 \end{cases} \tag{32}$$

A Perron-Frobenius type from Hanzon and Holland (2010b) result implies that $\sigma(\mathbf{A})$ contains a dominant real eigenvalue $\lambda_M < 0$ such that $\lambda_M = \max_{\lambda \in \sigma(A)} Re(\lambda)$. The Laplace transform of f is the proper rational function F

$$F(s) \; = \; \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \mathbf{d} \; = \; \frac{p(s)}{q(s)}, \tag{33}$$

where p and q are co-prime polynomials of degree n. As already mentioned a result from Lukacs (1970) states that an analytic Laplace Transform of an infinitely divisible probability density function cannot contain any zeros inside its strip of convergence. Letting $\Lambda(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \{s | \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \mathbf{d}\}$, then for an infinitely divisible probability density function with minimal realization $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ it must hold that $\max_{\lambda \in \Lambda(\mathbf{A},\mathbf{b},\mathbf{c},\mathbf{d})} Re(\lambda) \leq \lambda_M$. It follows since $\lambda_M < 0$ that $\Lambda(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \subset \mathbb{H}_-$ and therefore the F is of minimum phase.
As in Section 3 define $F = e^{-h}$ and it follows that $h' = -F'/F$. Hence by Feller (1971) F is the Laplace transform of an infinitely divisible distribution if and only if h' is completely monotonic on $[0, \infty)$ and $h(0) = 0$. It holds by construction that $h(0) = \log(F(0)) = 0$. Since F is a proper rational function and $\mathbf{d} > 0$, a well known identity gives

$$F^{-1}(s) \; = \; \mathbf{d}^{-1} - \mathbf{c}\mathbf{d}^{-1}(s\mathbf{I} - (\mathbf{A} - \mathbf{b}\mathbf{d}^{-1}\mathbf{c}))^{-1}\mathbf{b}\mathbf{d}^{-1}. \tag{34}$$

The zeros of F^{-1} correspond to the eigenvalues of $\mathbf{A}$ and the poles of $F^{-1}(s)$ are the eigenvalues of $\mathbf{A} - \mathbf{b}\mathbf{d}^{-1}\mathbf{c}$. Following a similar technique from Section 3 we have that
Lemma 5 *Let F be the minimum phase rational function $F(s) = \boldsymbol{c}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} + \boldsymbol{d}$ and $F = e^{-h}$. Then $h' = -F'/F$ is the Laplace transform of $Tr(e^{\boldsymbol{A}x}) - Tr(e^{\boldsymbol{B}_d x})$ where $\boldsymbol{B}_d = \boldsymbol{A} - \boldsymbol{b}\boldsymbol{d}^{-1}\boldsymbol{c}$*
Proof
Let $F = p/q$ where $p(s) = \mathbf{d}\det(s\mathbf{I} - \mathbf{B}_d)$ and $q(s) = \det(s\mathbf{I} - \mathbf{A})$. The remainder of the proof follows as in Lemma 3 ■.
Hence we conclude with
Theorem 2 *Given a minimal realization $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ that defines a not identically zero probability density function f defined as the mixture of an EPT function with a pointmass at zero. Assume the Laplace Transform F of f is of minimum phase. F is given by*

$$F(s) \; = \; \exp\Big(- \int_0^\infty (1 - e^{-sx}) \frac{Tr(e^{\boldsymbol{A}x}) - Tr(e^{\boldsymbol{B}_d x})}{x} dx \Big). \tag{35}$$

Then F is infinitely divisible iff $Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}_d x}) \geq 0$ for all $x \geq 0$

Comparing Eq. (41) to the Lévy-Khintchine formula in Eq. (2) we see that the corresponding Lévy triple for the density function with minimal realization $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ has $\sigma = 0$ indicating a pure jump process and Lévy measure

$$\nu(x) = \frac{Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}_d x})}{x} \quad , \quad x \geq 0 \tag{36}$$

Finally the drift component is

$$\begin{aligned} a &= -\int_0^1 x \, d\nu(x) \\ &= -\int_0^1 \Big(Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}_d x})\Big) dx \\ &= -Tr(\mathbf{A}^{-1}e^{\mathbf{A}}) + Tr(\mathbf{B}_d^{-1}e^{\mathbf{B}_d}) + Tr(\mathbf{A}^{-1}) - Tr(\mathbf{B}_d^{-1}) \ < \ \infty \end{aligned} \tag{37}$$

which is finite as $\{\sigma(\mathbf{A}) \cup \sigma(\mathbf{B}_d)\} \subset \mathbb{H}_-$. We see from Eq. (37) that the associated Lévy process is of finite variation which follows from Eq. (4).

The Budan-Fourier method of Hanzon and Holland (2010a) can be used to test for non-negativity of the ET function $Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}_d x})$ on a finite interval. As stated at the end of Section 3, results from Hanzon, Olivi & Sexton (2012) prove that if $\{\sigma(\mathbf{A}) \cup \sigma(\mathbf{b}_d)\}$ contains a unique dominant real pole λ_M then the Budan-Fourier technique can be used to locate all sign-changing zeros on the half line $[0, \infty)$.

Similar to Section 3 it is clear that if F is the Laplace Transform of an infinitely divisible EPT function and $\tilde{\lambda}_M = \max_{\lambda \in \sigma(\mathbf{B}_d)} Re(\lambda)$ then it must hold that $\tilde{\lambda}_M \leq \lambda_M$. This condition is equivalent to the result from Lukacs (1970) regarding the location of the poles and zeros of a rational Laplace Transform of an infinitely divisible distribution.

An identical argument to that given at the end of Section 3 in Eq. (25) implies the Laplace Transform of an infinitely divisible generalised EPT distribution can not contain more zeros than poles.

5. Infinitely Divisible Result for Rational Laplace Tranforms

In Sections 3 and 4 we made use of the fact that EPT and generalised EPT probability density functions, f, could be represented with the minimal realization $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ where $0 \leq \mathbf{d} < 1$. The case is now considered where the Laplace Transform of f is the rational function F given by

$$F(s) \;=\; \int_0^\infty e^{-sx} f(x)dx \;=\; \frac{p(s)}{q(s)},\; s > 0, \tag{38}$$

and $p,\; q$ are co-prime polynomials of orders m and n respectively where $m \leq n$. We denote $\Lambda(r(s)) = \{s | r(s) = 0\}$ as the zeros of the polynomial $r(s)$. As noted already, due to the Perron-Frobenius type result, F must contain a dominant real pole implying $\lambda_M \in \Lambda(q(s))$ where $\lambda_M = \max_{\lambda \in \Lambda(q(s))} Re(\lambda)$. Similarly, by the results of Lukacs (1970) a necessary condition for F to be infinitely divisible requires $\max_{\lambda \in \Lambda(p(s))} Re(\lambda) \leq \lambda_M$. Hence only probability density functions whose rational Laplace Transforms satisfy these conditions are considered.
Following the same method as in Section 3 by letting $F = e^{-h}$ it is known that F is infinitely divisible iff there exists a positive measure ν for which the Laplace transform of ν is $h' = -F'/F$. If $\Lambda(r(s)) \subset \mathbb{H}_-$ then from Lemma 2 for $s > 0$

$$\sum_{\lambda \in \Lambda(r(s))} \int_0^\infty e^{\lambda x} e^{-sx} dx \;=\; \frac{r'(s)}{r(s)}, \tag{39}$$

and using Lemma 3 we obtain the following Theorem.
Theorem 3 *Given a probability density function f with rational Laplace Transform $F = p/q$ where p and q are co-prime polynomials of orders m and n respectively such that $m \leq n$. If $\deg(p(s)) > 0$ and $\{\Lambda(p(s)) \cup \Lambda(q(s))\} \subset \mathbb{H}_-$. Then F is the Laplace transform of f given by*

$$F(s) \;=\; \exp\Big(- \int_0^\infty (1 - e^{-sx}) \frac{\sum_{\lambda \in \Lambda(q(s))} e^{\lambda x} - \sum_{\mu \in \Lambda(p(s))} e^{\mu x}}{x} dx \Big). \tag{40}$$

F is infinitely divisible iff $\sum_{\lambda \in \Lambda(q(s))} e^{\lambda x} \geq \sum_{\mu \in \Lambda(p(s))} e^{\mu x}$ for all $x \geq 0$
If $\deg(p(s)) = 0$ and the zeros of $\Lambda(q(s)) \subset \mathbb{H}_-$, then F is the Laplace transform of f given by

$$F(s) \;=\; \exp\Big(- \int_0^\infty (1 - e^{-sx}) \frac{\sum_{\lambda \in \Lambda(q(s))} e^{\lambda x}}{x} dx \Big). \tag{41}$$

F is infinitely divisible if $\sum_{\lambda \in \Lambda(q(s))} e^{\lambda x} \geq 0$ for all $x \geq 0$
The result in Theorem 3 coincide with the results obtained in Steutel (1967) for mixtures of exponential functions. Theorem 3 also clearly demonstrates that the number of zeros (including multiplicities) of $F(s)$ must be less than or equal to the number of poles (including multiplicities) of F if F is infinitely divisible.

6. Infinitely Divisible 2-EPT Distributions

We derive a sufficient condition for a generalised 2-EPT probability density function to be infinitely divisible. Consider the 2-EPT random variable X with generalised 2-EPT probability density function f, with realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P, \mathbf{d})$ given by

$$f(x) = \begin{cases} \mathbf{c}_N \; e^{\mathbf{A}_N \; x} \; \mathbf{b}_N & \text{if x} < 0 \\ 0 & \text{with probability } \mathbf{d} \\ \mathbf{c}_P \; e^{\mathbf{A}_P \; x} \; \mathbf{b}_P & \text{if x} > 0 \end{cases} \tag{42}$$

As f represents a probability density function, the Perron-Frobenius type result from Hanzon and Holland (2010a) implies $\lambda_{M_-} \in \sigma(\mathbf{A}_P)$ and $\lambda_{M_+} \in \sigma(\mathbf{A}_N)$ such that

$$\begin{aligned} \lambda_{M_-} &= \max_{\lambda \in \sigma(\mathbf{A}_P)} Re(\lambda) \\ \lambda_{M_+} &= \min_{\lambda \in \sigma(\mathbf{A}_N)} Re(\lambda). \end{aligned}$$

The Laplace Transform of f which is analytic for $s \in (\lambda_{M_-}, \lambda_{M_+})$ is given by

$$F(s) \;=\; -\mathbf{c}_N(s\mathbf{I} - \mathbf{A}_N)^{-1}\mathbf{b}_N + \mathbf{c}_P(s\mathbf{I} - \mathbf{A}_P)^{-1}\mathbf{b}_P + \mathbf{d} \;=\; \frac{p(s)}{q(s)}. \tag{43}$$

where p and q are co-prime polynomials of degree m and n respectively such that $m \leq n$.

The region $\mathcal{I}$ is defined such that $\mathcal{I} = \{u + iv | u \in (\lambda_{M_-}, \lambda_{M_+}), v \in \mathbb{R}\}$. Hence if F is the Laplace Transform of an infinitely divisible distribution then a result from Lukacs (1970) implies

$$\Lambda(p(s)) \cap \mathcal{I} \;=\; \emptyset. \tag{44}$$

Therefore if F is infinitely divisible then $F(is) \neq 0$ for all $s \in \mathbb{R}$. This property can be used to confirm that the Laplace Transform F is not infinitely divisible.
If F is infinitely divisible it can be factored as follows

$$\begin{aligned} F(s) &= F_1(s)\; F_2(s) \\ &= \frac{p_1(s)}{q_1(s)}\,\frac{p_2(s)}{q_2(s)}, \end{aligned}$$

such that F_1 and F_2 are both of minimum phase implying $\Lambda(p_1(s)) \cup \Lambda(q_1(s)) \subset \mathbb{H}_-$ and $\Lambda(p_2(s)) \cup \Lambda(q_2(s)) \subset \mathbb{H}_+$. We also see that $\max_{\lambda \in \Lambda(p_1(s))} Re(\lambda) \leq \lambda_{M_-}$ and $\min_{\lambda \in \Lambda(p_2(s))} Re(\lambda) \geq \lambda_{M_-}$.

A sufficient condition that F is infinitely divisible is if both F_1 and F_2 are infinitely divisible. Using the techniques of Sections 3 and 4 the Lévy Triples of F_1 and F_2 can be determined as (a_1, σ_1, ν_1) and (a_2, σ_2, ν_2) respectively. Scaling would also be necessary such that $F_1(0) = F_2(0) = 1$. From the results of Sections 3 and 4 it is known that F_1 can not be infinitely divisible if $\deg(p_1) > \deg(q_1)$ (i.e. F_1 cannot contain more zeros than poles). The same result applies to F_2. Hence for F_1 or F_2 to be infinitely divisible it is necessary that they are proper (or strictly proper) rational functions.

If F_1 and F_2 are both infinity divisible then the Lévy triple (a, σ, ν) for F can be determined and is easily seen to be given by $a = a_1 + a_2$, $\sigma^2 = \sigma_1^2 + \sigma_2^2$ and

$$\nu(x) = \begin{cases} \nu_1(x) & \text{if x} > 0 \\ \\ \nu_2(x) & \text{if x} < 0 \end{cases} \tag{45}$$

7. Variance Gamma Example

Consider a random variable X with a Variance Gamma probability density function f which has input parameters (C, G, M) where "C" is integer and $G, M > 0$. The Laplace transform F of the density function is a rational and analytic for $s \in (-G, M)$

$$F(s) = \Big(\frac{MG}{MG + (M-G)s - s^2}\Big)^C \tag{46}$$

$$= \Big(\frac{-M}{s-M}\Big)^C \Big(\frac{G}{s+G}\Big)^C \tag{47}$$

$$= F_1(s)\ F_2(s). \tag{48}$$

$F(is)$ is the strictly proper rational characteristic function of X. F can be represented in state space form as

$$\Big(\frac{-M}{s-M}\Big)^C \Big(\frac{G}{s+G}\Big)^C = \mathbf{c}_N(s\mathbf{I} - \mathbf{A}_N)^{-1}\mathbf{b}_N\ \mathbf{c}_P(s\mathbf{I} - \mathbf{A}_P)^{-1}\mathbf{b}_P \tag{49}$$

where $\mathbf{c}_P$ and $\mathbf{c}_N$ are $1 \times C$ row vectors

$$\mathbf{c}_N = (0, 0, ..., 0, -M^C) \quad , \quad \mathbf{c}_P = (0, 0, ..., 0, G^C) \tag{50}$$

Likewise we have $\mathbf{b}_P = \mathbf{b}_N = (1, 0, 0, ..., 0)^T$ while $\mathbf{A}_N$ and $\mathbf{A}_P$ are square $C \times C$ matrices given by

$$\mathbf{A}_N = \begin{pmatrix} M & 0 & 0 & ... & 0 \\ -1 & M & 0 & ... & 0 \\ 0 & -1 & M & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & ... & 0 & -1 & M \end{pmatrix} \quad , \quad \mathbf{A}_P = \begin{pmatrix} -G & 0 & 0 & ... & 0 \\ 1 & -G & 0 & ... & 0 \\ 0 & 1 & -G & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & ... & 0 & 1 & -G \end{pmatrix} \tag{51}$$

It is clear from Eq. (46) that both F_1 and F_2 are strictly proper rational functions. Hence the results of Section 3 can be used to determine if F_1 and F_2 are the Laplace Transforms of infinitely divisible EPT distributions with support on $\mathbb{H}_-$ and $\mathbb{H}_+$ respectively. It should be noted that

$$\deg\Big(\mathbf{c}_N(s\mathbf{I} - \mathbf{A}_N)^*\mathbf{b}_N\Big) = \deg\Big(\mathbf{c}_P(s\mathbf{I} - \mathbf{A}_P)^*\mathbf{b}_P\Big) = 0, \tag{52}$$

From Theorem 1 we can conclude that F_2 is the Laplace Transform of an infinitely if and only if

$$Tr(e^{\mathbf{A}_P x}) \geq 0, \tag{53}$$

for all $x > 0$. It is clear that

$$Tr(e^{\mathbf{A}_P x}) = Ce^{-Gx} \geq 0 \tag{54}$$

for all $x > 0$ so we conclude that F_2 is indeed the Laplace Transform of an infinitely divisible distribution. We also conclude that F_2 is infinitely divisible as

$$Tr(e^{\mathbf{A}_N x}) \;=\; Ce^{Mx} \geq 0 \tag{55}$$

for all $x < 0$. Hence we conclude that the 2-EPT random variable X is infinitely divisible. The Levy triple associated with X is given by (a, σ, ν) where $\sigma = 0$,

$$a \;=\; -\int_0^1 Ce^{-Gx}dx - \int_{-1}^0 Ce^{Mx}dx \tag{56}$$

$$\;=\; \frac{C}{G}\Big(e^{-G} - 1\Big) \;+\; \frac{C}{M}\Big(1 - e^{-M}\Big) \tag{57}$$

and

$$\nu(x) = \begin{cases} Ce^{Mx} & \text{if x} < 0 \\ \\ Ce^{-Gx} & \text{if x} > 0 \end{cases} \tag{58}$$

8. Conclusion

Using a result from Feller (1971) in conjunction with Bernsteins Theorem from Widder (1941) it is illustrated how to characterise infinitely divisible EPT functions. Using a similar technique it is also possible to establish if an EPT function mixed a pointmass at zero is infinitely divisible. 2-EPT probability density function were introduced in Hanzon and Sexton (2012) and by factorizing the characteristic function of the 2-EPT density into two “minimum phase” rational functions a sufficient condition is provided to determine if the density is infinitely divisible. Also, formulae are given for the Lévy triple of an infinitely divisible 2-EPT / EPT density function. An example included illustrates that the Variance Gamma distribution (under a parameter restriction) is infinitely divisible and its Lévy triple is derived under this restriction.

Each infinitely divisible EPT / 2-EPT density function generates a 2-EPT / EPT Lévy process whose triple can be derived. In Hanzon and Sexton (2012) it is shown how these 2-EPT Lévy processes can be used to model an assets log-returns over a fixed period τ. Closed form European Option pricing formulae can then be derived for such options with integer multiples of τ to maturity. It is possible to derive closed form formulae for Lookback Options with fixed and floating strikes. The Greeks of all these options can also be derived.

Typically numerical algorithms have been used to derive Option Prices and their Greeks when the underlying has log-returns being modelled with a Lévy process.

References

Delbaen, F., Schachermayer, W., *A General Version of the Fundamental Theorem of Asset Pricing*, Mathematische Annalen, Vol. 300, pages 463-520, 1994

Feller, W. *An Introduction to Probability Theory and its Applications, Vol. II* John Wiley and sons, 1971

Widder, D.V. *The Laplace Transforms* Princeton University Press, 1941

Madan, D.B. *Option Pricing, Interest Rates and Risk Management: Purely Discontinuous Asset Price Processes* Chapter 4, Cambridge University Press, 2001

Kailath, T., *Linear Systems* Prentice Hall, 1980

Hanzon, B., Holland, F. *Non-Negativity of Exponential Polynomial Trigonometric Functions - A Budan Fourier Sequence Approach* "www.2-ept.com", BFS Toronto Poster 429, June 22 - 26, 2010

Hanzon, B., Holland, F. *Non-negativity analysis for Exponential-Polynomial-Trigonometric Functions on the non-negative real half-line* To Appear. Draft version available online at "www.edgeworth.biz", 2011

Sexton, C., Olivi, M., Hanzon, B., *Rational Approximation of Transfer Functions for Non-Negative EPT Densities*, Conference Proceedings, SYSID 2012. (Draft version available on "www.2-ept.com".)

Lukacs, E.,*Characteristic Functions 2nd Edition*, Griffen, London, 1970

Hanzon, B., Sexton, C., *State Space Calculations for two-sided EPT Densities with Financial Modelling Applications.* Submitted. Draft version available at "www.2-ept.com", 2012

Kyprianou A.E., Loeffen, R., *Lévy Processes in Finance Distinguished by their Coarse and Fine Path Properties* Exoticc Option Pricing and Advanced Lévy Models, John Wiley and Sons, 2005

Cont, R., Tankov, P.*Financial Modelling with Jump Processes*,Chapman & Hall, 2003

Carr, P., Madan, D., Chang, E., *The Variance Gamma Process and Option Pricing* European Finance Review 2: 79 - 105, 1998

Steutel F.W., *Note on the infinite divisibility of Exponential Mixtures*,The Annals of Mathematical Statistics, Vol. 38, No. 4 (Aug., 1967), pp. 1303-1305, 1967

Hanzon, B., Holland, F. *On a Perron Frobenius type result for Non-Negative Impulse Response Functions* "http://euclid.ucc.ie/pages/staff/hanzon/HanzonHollandDominantPoleLisbonPaperisn.pdf"