Approximating EPT Densities via RARL2 with Financial Modelling Aplications of two sided EPT Densities

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INTRODUCTION

EPT Functions on Half Line

$$f(x) \;=\; Re\Big(\sum_{k=1}^{K} \; p_k(x)\; e^{\mu_k x}\Big) \qquad\qquad x \geq 0$$

$$f(x) \;=\; \mathbf{c}e^{\mathbf{A}x}\mathbf{b} \qquad\qquad if\ x \;\geq\; 0$$

$\mathbf{A}$ is an $(n \times n)$ matrix, $\mathbf{c}$ a $(1 \times n)$ row vector and $\mathbf{b}$ a $(n \times 1)$ column vector

- Class contains the real polynomials, real exponential functions and scaled real trigonometric polynomials
- Closed under addition and multiplication
- Strictly proper rational characteristic function

$$\phi(iu) = \int_0^{\infty} e^{iux}\mathbf{c}e^{\mathbf{A}x}\mathbf{b}dx = -\mathbf{c}(\mathbf{I}iu + \mathbf{A})^{-1}\mathbf{b} = \frac{p(iu)}{q(iu)}$$

q is a polynomial of degree n while p is a polynomial of degree $m < n$

ϕ a rational function of McMillan degree n

EPT Density

Triple $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ such that

$$f(x) = \mathbf{c}e^{\mathbf{A}x}\mathbf{b} \geq 0 \qquad \forall\, x \geq 0$$

- Test for non-negativity, Hanzon and Holland (2010)
- We use Minimal Realizations (use a minimal realization algorithm if necessary)
- Stability of $\mathbf{A}$ as eigenvalues located in open left plane

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \mathbf{c}e^{\mathbf{A}x}\mathbf{b} = 0$$

- Normalization due to finite integrability

$$\int_0^\infty \mathbf{c}e^{\mathbf{A}x}\mathbf{b}dx = -\mathbf{c}\mathbf{A}^{-1}\mathbf{b} < \infty$$

Triple $(\mathbf{A}, \mathbf{b}, \tilde{\mathbf{c}})$ now represents a probability density function

$$\tilde{\mathbf{c}} = \frac{\mathbf{c}}{-\mathbf{c}\mathbf{A}^{-1}\mathbf{b}}$$

2-EPT Density

A 2-EPT density requires two EPT densities defined on either half real line

$$f(x) = \begin{cases} \mathbf{c}_N \; e^{\mathbf{A}_N \; x} \; \mathbf{b}_N & \text{if} \;\; x \leq 0 \\ \\ \mathbf{c}_P \; e^{\mathbf{A}_P \; x} \; \mathbf{b}_P & \text{if} \;\; x > 0 \end{cases} \qquad (1)$$

$\mathbf{A}_N$ is “anti-stable” with all eigenvalues located in open right half plane

$$\lim_{x \to \infty} \mathbf{c}_P e^{\mathbf{A}_P x} \mathbf{b}_P = \lim_{x \to -\infty} \mathbf{c}_N e^{\mathbf{A}_N x} \mathbf{b}_N = 0$$

Finite integrability allows for normalization

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{0} \mathbf{c}_N e^{\mathbf{A}_N x} \mathbf{b}_N \; dx + \int_{0}^{\infty} \mathbf{c}_P e^{\mathbf{A}_P \; x} \mathbf{b}_P \; dx = \mathbf{c}_N \mathbf{A}_N^{-1} \mathbf{b}_N - \mathbf{c}_P \mathbf{A}_P^{-1} \mathbf{b}_P$$

Triples $(\mathbf{A}_P, \mathbf{b}_P, \tilde{\mathbf{c}}_P)$ and $(\mathbf{A}_N, \mathbf{b}_N, \tilde{\mathbf{c}}_N)$ define probability density function where $\tilde{\mathbf{c}_P}$ and $\tilde{\mathbf{c}}_N$ are scaled ensure density integrates to unity.

State Space Representation

Rational Characteristic function decomposed

$$\begin{aligned}\phi(iu) &= \int_{-\infty}^{0} \mathbf{c}_N\, e^{\mathbf{A}_N\, x}\, \mathbf{b}_N\, e^{iu\,x}\, dx + \int_{0}^{\infty} \mathbf{c}_P\, e^{\mathbf{A}_P\, x}\, \mathbf{b}_P\, e^{iu\,x}\, dx \\ &= \mathbf{c}_N\,(\mathbf{I}iu + \mathbf{A}_N)^{-1}\,\mathbf{b}_N - \mathbf{c}_P\,(\mathbf{I}iu + \mathbf{A}_P)^{-1}\,\mathbf{b}_P \\ &= \phi_N(iu) + \phi_P(iu) = \frac{p(iu)}{q(iu)}\end{aligned} \tag{2}$$

Represented in state space form

$$\phi(u) = \pi\left(\begin{array}{cc|c} \mathbf{A}_P & 0 & \mathbf{b}_P \\ 0 & \mathbf{A}_N & \mathbf{b}_N \\ \hline -\mathbf{c}_P & \mathbf{c}_N & 0 \end{array}\right) \tag{3}$$

Numerous operations can be carried out when in this form using state space techniques of Hanzon and Ober (2001), Hanzon and Scherrer (2008)

Point Mass at Zero

A natural generalisation of the EPT class also allows for a pointmass at zero

$$f(x) + d\delta \qquad 0 \leq d \leq 1$$

where δ is the delta distribution and $f(x)$ a 2-EPT density function

Hence the state space representation is given by

$$\phi(u) \;=\; \pi \left(\begin{array}{cc|c} \mathbf{A}_P & 0 & \mathbf{b}_P \\ 0 & \mathbf{A}_N & \mathbf{b}_N \\ \hline -\mathbf{c}_P & \mathbf{c}_N & d \end{array} \right)$$

which is the rational characteristic function

$$\phi(u) \;=\; \mathbf{c}_N \left(\mathbf{I}iu \;+\; \mathbf{A}_N\right)^{-1} \mathbf{b}_N \;-\; \mathbf{c}_P \left(\mathbf{I}iu \;+\; \mathbf{A}_P\right)^{-1} \mathbf{b}_P \;+\; d$$

Phase-Type, Matrix Exponential and Positive Realizations

Positive Realizations are strict subset of EPT class

- Non-negativity guaranteed
- Positive realization may not exist
- If positive realization does exist, dimensions may be greater than McMillan degree

Similar arguments apply to phase type distributions where a Phase-Type distribution may not exist for a given characteristic function.

Matrix Exponential class is equivalent to EPT class

- Realization could be found from rational characteristic function using companion matrix
- No reference to minimal realizations in literature
- Matrix Exponential only defined on non-negative half line

OPERATIONS

Scaling

Let X be a 2-EPT random variable with density function given by Eq. (1). Let $\alpha \in \mathbb{R}^1$ such that $\alpha > 0$ and $Y = \alpha X$.

$$\begin{aligned} G(y_0) &= \mathbb{P}(Y \leq y_0) \\ &= \mathbb{P}(\alpha X \leq y_0) \\ &= \mathbb{P}(X \leq \frac{y_0}{\alpha}) \\ &= F(\frac{y_0}{\alpha}) \end{aligned}$$

Differentiating we get

$$g(y_0) = f(\frac{y_0}{\alpha})\frac{1}{\alpha}$$

and leaving $x_0 = \frac{y_0}{\alpha}$ we obtain that

$$\alpha\, g(\alpha\, x_0) = f(x_0)$$

Hence if our original triples for X were $(\mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$ and $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N)$ the associated triples for Y are given by $(\alpha\mathbf{A}_P, \mathbf{b}_P, \alpha\mathbf{c}_P)$ and $(\alpha\mathbf{A}_N, \mathbf{b}_N, \alpha\mathbf{c}_N)$

Translation

Let X be a random variable with a 2-EPT density function, f, as in Eq. (1). Let $x_0 \in \mathbb{R}^1$. The random variable $X + x_0$ has a density function $g(x)$ such that $g(x) \; = \; f(x - x_0)$.

$$g(x) = f(x - x_0) = \begin{cases} \mathbf{c}_N \; e^{\mathbf{A}_N x} \; e^{-\mathbf{A}_N x_0} \mathbf{b}_N & \text{if } \; x \leq x_0 \\ \mathbf{c}_P \; e^{\mathbf{A}_P x} \; e^{-\mathbf{A}_P x_0} \mathbf{b}_P & \text{if } \; x > x_0 \end{cases}$$

$$\begin{aligned} \phi(iu) &= \int_{-\infty}^{x_0} \mathbf{c}_N \; e^{\mathbf{A}_N \, x} \; e^{-\mathbf{A}_N x_0} \; e^{iu \, x} \; \mathbf{b}_N \; dx + \int_{x_0}^{\infty} \mathbf{c}_P \; e^{\mathbf{A}_P \, x} \; e^{-\mathbf{A}_P x_0} \; e^{iu \, x} \; \mathbf{b}_P \; dx \\ &= \mathbf{c}_N (\mathbf{A}_N + ui\mathbf{I})^{-1} e^{(\mathbf{A}_N + ui\mathbf{I}) x_0} e^{-\mathbf{A}_N x_0} \mathbf{b}_N - \mathbf{c}_P (\mathbf{A}_P + ui\mathbf{I})^{-1} e^{(\mathbf{A}_P + ui\mathbf{I}) x_0} e^{-\mathbf{A}_P x_0} \mathbf{b}_P \end{aligned}$$

which is not rational.

We do not consider this operation further as it does not fit into the EPT class

Sums of 2-EPT Functions

Consider the mixture density $g(x)$ of two 2-EPT density functions $f_1(x)$ and $f_2(x)$ as defined in Eq. (1) with weights w and $(1-w)$ resp. where $0 < w < 1$.

$$g(x) = \begin{cases} w\,\mathbf{c}_{N,1}\, e^{\mathbf{A}_{N,1}\,x}\mathbf{b}_{N,1} \;+\; (1-w)\mathbf{c}_{N,2}\, e^{\mathbf{A}_{N,2}\,x}\mathbf{b}_{N,2} = \mathbf{c}_N\, e^{\mathbf{A}_N\,x}\mathbf{b}_N & \text{if } x \le 0 \\ w\,\mathbf{c}_{P,1}\, e^{\mathbf{A}_{P,1}\,x}\mathbf{b}_{P,1} \;+\; (1-w)\mathbf{c}_{P,2}\, e^{\mathbf{A}_{P,2}\,x}\mathbf{b}_{P,2} = \mathbf{c}_P\, e^{\mathbf{A}_P\,x}\mathbf{b}_P & \text{if } x > 0 \end{cases}$$

For $x > 0$ we have $\mathbf{c}_P = (w\,\mathbf{c}_{P,1},\ (1-w)\mathbf{c}_{P,2})$, $\mathbf{b}_P = (\mathbf{b}_{P,1};\ \mathbf{b}_{P,2})$
We split the **A** matrices into a block diagonal structure as follows

$$\mathbf{A}_N = \begin{pmatrix} \mathbf{A}_{N,1} & 0 \\ 0 & \mathbf{A}_{N,2} \end{pmatrix} \quad , \quad \mathbf{A}_P = \begin{pmatrix} \mathbf{A}_{P,1} & 0 \\ 0 & \mathbf{A}_{P,2} \end{pmatrix}$$

If we allow $w < 0$ then we must test for non-negativity

Product of two 2-EPT Densities

To calculate the product of two 2-EPT probability densities $f_1(x)$ and $f_2(x)$

$$f_i(x) = \begin{cases} \mathbf{c}_{N,i}\ e^{\mathbf{A}_{N,i}\ x}\ \mathbf{b}_{N,i} & \text{if } \ x \leq 0 \\ \mathbf{c}_{P,i}\ e^{\mathbf{A}_{P,i}\ x}\ \mathbf{b}_{P,i} & \text{if } \ x > 0 \end{cases}$$

where $\mathbf{A}_{N,i}$ and $\mathbf{A}_{P,i}$ are square matrices of size $n_{N,i}$ and $n_{P,i}$ respectively.

$$g(x) \ = \ f_1(x)\ f_2(x) \ = \begin{cases} \mathbf{c}_{N,1}\ e^{\mathbf{A}_{N,1}\ x}\ \mathbf{b}_{N,1}\ \mathbf{c}_{N,2}\ e^{\mathbf{A}_{N,2}\ x}\ \mathbf{b}_{N,2} \ = \ \mathbf{c}_{N}\ e^{\mathbf{A}_{N}\ x}\ \mathbf{b}_{N} & \text{if } \ x \leq 0 \\ \mathbf{c}_{P,1}\ e^{\mathbf{A}_{P,1}\ x}\ \mathbf{b}_{P,1}\ \mathbf{c}_{P,2}\ e^{\mathbf{A}_{P,2}\ x}\ \mathbf{b}_{P,2} \ = \ \mathbf{c}_{P}\ e^{\mathbf{A}_{P}\ x}\ \mathbf{b}_{P} & \text{if } \ x > 0 \end{cases}$$

For $x \ > \ 0$ we obtain the triple $(\mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$

$$\begin{aligned} \mathbf{A}_P &= \mathbf{A}_{P,1} \otimes \mathbf{I}_{n_{P,2}} + \mathbf{I}_{n_{P,1}} \otimes \mathbf{A}_{P,2} \\ \mathbf{b}_P &= \mathbf{b}_{P,1} \otimes \mathbf{b}_{P,2} \\ \mathbf{c}_P &= \mathbf{c}_{P,1} \otimes \mathbf{c}_{P,2} \end{aligned}$$

where I_n is the $n \times n$ identity matrix.

Moment Calculations via Characteristic Function

The characteristic function of a 2-EPT density is

$$\phi(u) = \mathbf{c}_N \left(\mathbf{I}u + \mathbf{A}_N\right)^{-1} \mathbf{b}_N \; - \; \mathbf{c}_P \left(\mathbf{I}u + \mathbf{A}_P\right)^{-1} \mathbf{b}_P$$

The k^{th} derivative of ϕ is again a rational functions with the same pole locations as ϕ

$$\frac{d^k\phi(u)}{du^k} = (-1)^k k!\mathbf{c}_N(\mathbf{I}u + \mathbf{A}_N)^{-(k+1)}\mathbf{b}_N - (-1)^k k!\mathbf{c}_P(\mathbf{I}u + \mathbf{A}_P)^{-(k+1)}\mathbf{b}_P$$

Evaluated at $u = 0$ we get

$$\left.\frac{d^k\phi(u)}{du^k}\right|_{u=0} = (-1)^k k!\mathbf{c}_N(\mathbf{A}_N)^{-(k+1)}\mathbf{b}_N - (-1)^k k!\mathbf{c}_P(\mathbf{A}_P)^{-(k+1)}\mathbf{b}_P$$

All moments must exist

Calculating Moments using Direct Integration

Calculating the k^{th} moment by integrating $x^k f(x)$ over the relevant domain. It must be noted that $p(x) = x^k = \mathbf{c}_k e^{\mathbf{A}_k x}\mathbf{b}_k$ where $\mathbf{A}_k$ is a $(k+1)$ square nilpotent matrix.

$$\begin{aligned}
\int_{-d}^{d} x^k f(x)dx &= \int_{-d}^{0} \mathbf{c}_k e^{\mathbf{A}_k x}\mathbf{b}_k \, \mathbf{c}_N e^{\mathbf{A}_N x}\mathbf{b}_N dx + \int_{0}^{d} \mathbf{c}_k e^{\mathbf{A}_k x}\mathbf{b}_k \, \mathbf{c}_P e^{\mathbf{A}_P x}\mathbf{b}_P dx \\
&= \int_{-d}^{0} \tilde{\mathbf{c}}_N \, e^{\tilde{\mathbf{A}}_N \, x}\tilde{\mathbf{b}}_N \, dx + \int_{0}^{d} \tilde{\mathbf{c}}_P \, e^{\tilde{\mathbf{A}}_P \, x}\tilde{\mathbf{b}}_P \, dx \\
&= \tilde{\mathbf{c}}_N \tilde{\mathbf{A}}_N^{-1} \tilde{\mathbf{b}}_N - \tilde{\mathbf{c}}_N \tilde{\mathbf{A}}_N^{-1} e^{-\tilde{\mathbf{A}}_N d}\tilde{\mathbf{b}}_N + \tilde{\mathbf{c}}_P \tilde{\mathbf{A}}_P^{-1} e^{\tilde{\mathbf{A}}_P d}\tilde{\mathbf{b}}_P - \tilde{\mathbf{c}}_P \tilde{\mathbf{A}}_P^{-1}\tilde{\mathbf{b}}_P
\end{aligned}$$

where the triples $(\tilde{\mathbf{A}}_P, \ \tilde{\mathbf{b}}_P, \ \tilde{\mathbf{c}}_P)$ and $(\tilde{\mathbf{A}}_N, \ \tilde{\mathbf{b}}_N, \ \tilde{\mathbf{c}}_N)$ can be determined using the product formulae

In the limit $d \to \infty$ we obtain

$$\int_{-\infty}^{\infty} x^k \, \mathbf{c} \, e^{\mathbf{A} \, x}\mathbf{b} \, dx = \tilde{\mathbf{c}}_N \, \tilde{\mathbf{A}}_N^{-1} \, \tilde{\mathbf{b}}_N - \tilde{\mathbf{c}}_P \, \tilde{\mathbf{A}}_P^{-1} \, \tilde{\mathbf{b}}_P$$

Minimum of 2-EPT Random Variables

Consider two independent 2-EPT random variables X, Y with densities f_i for $i = 1, 2$. The following Cumulative Distribution Functions (CDF) are derived

$$F_i(z) = \begin{cases} \mathbf{c}_{N,i}\ \mathbf{A}_{N,i}^{-1} e^{\mathbf{A}_{N,i}\ z}\ \mathbf{b}_{N,i} & \text{if } z \leq 0 \\ \mathbf{c}_{P,i}\ \mathbf{A}_{P,i}^{-1} e^{\mathbf{A}_{P,i}\ z}\ \mathbf{b}_{P,i} + 1 & \text{if } z > 0 \end{cases}$$

If we let $W = Min(X, Y)$, using the CDFs and the property

$$\mathbb{P}[W \leq w] = 1 - \mathbb{P}[X \geq w]\mathbb{P}[Y \geq w] \quad \ldots \quad \textit{by independence}$$

we obtain the density function $h(w)$ for the minimum

$$h(w) = \begin{cases} \mathbf{c}_{N,1} e^{\mathbf{A}_{N,1}\ w} \mathbf{b}_{N,1} + \mathbf{c}_{N,2} e^{\mathbf{A}_{N,2}\ w} \mathbf{b}_{N,2} - \mathbf{c}_N \tilde{\mathbf{A}}_{N,1,N,2}^{-1} \mathbf{A}_{N,1,N,2} e^{\tilde{\mathbf{A}}_{N,1,N,2} w} \mathbf{b}_N & \text{w} \leq 0 \\ -\mathbf{c}_P \tilde{\mathbf{A}}_{P,1,P,2}^{-1} \mathbf{A}_{P,1,P,2} e^{\tilde{\mathbf{A}}_{P,1,P,2} w} \mathbf{b}_P & \text{w} > 0 \end{cases}$$

A similar result can be found for $Z = Max(X, Y)$

Convolution of EPT Densities

Given two independent 2-EPT random variables X, Y with triples $(\mathbf{A}_1, \mathbf{b}_1, \mathbf{c}_1)$ and $(\mathbf{A}_2, \mathbf{b}_2, \mathbf{c}_2)$ respectively. The state space representation of $Z = X + Y$ is

$$\phi_Z(u) = \phi_X(u)\phi_Y(u) = \pi\left(\begin{array}{cc|c} \mathbf{A}_1 & \mathbf{b}_1\mathbf{c}_2 & \mathbf{b}_2 \\ 0 & \mathbf{A}_2 & 0 \\ \hline \mathbf{c}_1 & 0 & 0 \end{array}\right) = \pi\left(\begin{array}{c|c} \mathbf{A} & \mathbf{b} \\ \hline \mathbf{c} & 0 \end{array}\right) \qquad (4)$$

where

$$\mathbf{A}_1 = \begin{pmatrix} \mathbf{A}_{P,1} & 0 \\ 0 & \mathbf{A}_{N,1} \end{pmatrix} \quad , \quad \mathbf{A}_2 = \begin{pmatrix} \mathbf{A}_{P,2} & 0 \\ 0 & \mathbf{A}_{N,2} \end{pmatrix}$$

To ensure the rational characteristic function of Z in Eq. (4) can be decomposed appropriately we must block diagonalise $\mathbf{A}$ and separate the eigenvalues according to the sign of their real part

Block Diagonalise **A** while separating the Eigenvalues

A basis transformation is used to separate the eigenvalues of **A**. Assuming the sequence of transformations can be expressed by **Q** we obtain

$$\phi_Z(u) = \pi \left(\begin{array}{c|c} \mathbf{QAQ}^{-1} & \mathbf{Qb} \\ \hline \mathbf{cQ}^{-1} & 0 \end{array} \right) \quad , \quad \textit{where} \;\; \mathbf{QAQ}^{-1} = \begin{pmatrix} \mathcal{A}_P & \mathcal{V} \\ 0 & \mathcal{A}_N \end{pmatrix}$$

A transformation with T is used to block diagonalise $\mathbf{QAQ}^{-1}$

$$\mathbf{TQAQ}^{-1}\mathbf{T}^{-1} = \begin{pmatrix} \mathcal{A}_P & \mathcal{T}\mathcal{A}_P - \mathcal{A}_N\mathcal{T} + \mathcal{V} \\ 0 & \mathcal{A}_N \end{pmatrix} \quad , \quad \textit{where} \;\; \mathbf{T} = \begin{pmatrix} \mathbf{I} & \mathcal{T} \\ 0 & \mathbf{I} \end{pmatrix}$$

$\mathcal{T}$ is chosen as the solution to the sylvester equation

$$\mathcal{T}\mathcal{A}_P - \mathcal{A}_N\mathcal{T} + \mathcal{V} = 0$$

The original triple $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is now given by $(\mathbf{TQAQ}^{-1}\mathbf{T}^{-1},\ \mathbf{TQb},\ \mathbf{cQ}^{-1}\mathbf{T}^{-1})$

Composition of Rational Functions

Working in the broader class of 2-EPT densities where the pointmass at zero is included we consider the composition of two rational functions, G, with minimal realization $(\mathbf{A}_1, \mathbf{b}_1, \mathbf{c}_1, d_1)$ and, ϕ, with minimal realization $(\mathbf{A}_2, \mathbf{b}_2, \mathbf{c}_2, d_2)$.

$$G(\phi(u)) = d_1 + \mathbf{c}_1\left(\frac{1}{\phi(u)} - \mathbf{A}_1\right)^{-1}\mathbf{b}_1$$

where

$$\begin{aligned} \phi(u) &= \mathbf{c}_2(u\mathbf{I} - \mathbf{A}_2)^{-1}\mathbf{b}_2 + d_2 \\ (\phi(u))^{-1} &= d_2^{-1} - d_2^{-1}\mathbf{c}_2(u\mathbf{I} - \mathbf{A}_2 + \mathbf{b}_2\, d_2^{-1}\mathbf{c}_2)^{-1}\mathbf{b}_2 d_2^{-1} \end{aligned}$$

Implementing Proposition (3.2) from Hanzon and Scherrer (2008) we have that $G(\phi(u))$ is a rational function with realization

$$\begin{aligned} \mathbf{A} &= \mathbf{I}_{n_1} \otimes \tilde{\mathbf{A}_2} + (\mathbf{A}_1 - \tilde{d_2}\mathbf{I}_{n_1})^{-1} \otimes \tilde{\mathbf{b}_2}\tilde{\mathbf{c}_2} \\ \mathbf{b} &= -(\mathbf{A}_1 - \tilde{d_2}\mathbf{I}_{n_1})^{-1}\mathbf{b}_1 \otimes \tilde{\mathbf{b}_2} \\ \mathbf{c} &= \mathbf{c}_1(\mathbf{A}_1 - \tilde{d_2}\mathbf{I}_{n_1})^{-1} \otimes \tilde{\mathbf{c}_2} \\ d &= d_1 - \mathbf{c}_1(\mathbf{A}_1 - \tilde{d_2}\mathbf{I}_{n_1})^{-1}\mathbf{b}_1 \end{aligned}$$

where $\tilde{d_2}$ is not an eigenvalue of $\mathbf{A}_1$

A 2-EPT Approach to Financial Modelling with Variance Gamma

Variance Gamma Density

$$f_{VG}(x; C, G, M) = \frac{(GM)^C}{\sqrt{\pi}\,\Gamma(C)}\, exp\Big(\frac{(G-M)x}{2}\Big)\Big(\frac{|x|}{G+M}\Big)^{C-1/2} K_{C-1/2}\Big(\frac{(G+M)|x|}{2}\Big)$$

Variance Gamma (VG) Characteristic Function

$$\phi(ui; C, G, M) \;=\; \Big(\frac{G\ M}{G\ M + (M\ -\ G)iu + u^2}\Big)^C$$

For integer "C" the characteristic function is rational and the density simplifies to

$$f_{VG}(x; C, G, M) = exp\Big(\frac{(G-M)x-(G+M)|x|}{2}\Big) \underbrace{\frac{(MG)^C}{(C-1)!}\sum_{k=0}^{C-1}\frac{(C-1+k)!(G+M)^{-C-k}|x|^{C-1-k}}{(C-1-k)!\ k!}}_{p(x)}$$

Variance Gamma Density

Splitting the density around the origin

$$f_{VG}(x; C, G, M) = \begin{cases} exp(G\ x)\frac{(MG)^C}{(C-1)!}\sum_{s=0}^{C-1}\frac{(2(C-1)-s)!(G+M)^{-2C+1+s}|x|^s}{s!\ (C-1-s)!} & \text{if } \ x \leq 0 \\ exp(-M\ x)\frac{(MG)^C}{(C-1)!}\sum_{s=0}^{C-1}\frac{(2(C-1)-s)!(G+M)^{-2C+1+s}|x|^s}{s!\ (C-1-s)!} & \text{if } \ x > 0 \end{cases}$$

Polynomial part equal for all x implying $\mathbf{c}_P = \mathbf{c}_N = \mathbf{c}$ where entries are

$$c_s = \frac{(MG)^C}{(C-1)!}\frac{(2(C-1)-s)!(G+M)^{-2C+1+s}}{(C-1-s)!} \qquad \ldots \qquad s \in \{0, 1, ..., C-1\}$$

Similarly $\mathbf{b}_P = \mathbf{b}_N = \mathbf{b} = (1, 0, ..., 0)^T$ and let $\mathbf{a}$ be a $C \times C$ nilpotent matrix

$$f_{VG}(x; C, G, M) = \begin{cases} \mathbf{c}e^{Gx}e^{-\mathbf{a}x}\mathbf{b} = \mathbf{c}e^{(G\mathbf{I}-\mathbf{a})x}\mathbf{b} = \mathbf{c}e^{\mathbf{A}_N x}\mathbf{b} & \text{if } \ x \leq 0 \\ \mathbf{c}e^{-Mx}e^{\mathbf{a}x}\mathbf{b} = \mathbf{c}e^{(-M\mathbf{I}+\mathbf{a})x}\mathbf{b} = \mathbf{c}e^{\mathbf{A}_P x}\mathbf{b} & \text{if } \ x > 0 \end{cases}$$

Variance Gamma Price Process

The Variance Gamma Process as described in Carr, Ghang, Madan (1998) under the risk neutral measure $\mathbb{Q}$

$$S_T \;=\; S_t \; e^{(\omega + r)(T-t) \,+\, x_{T-t}}$$

To ensure $\mathbb{E}_{\mathbb{Q}}[S_T] \;=\; S_t \; e^{r\tau}$, where $\tau = T - t$

$$\omega \;=\; -C \; ln\Big(\frac{MG}{MG \,+\, (M-G) \,-\, 1}\Big)$$

Hence the shifted log return is given by

$$ln(\frac{S_T}{S_t}) \;-\; (\omega + r)\tau \;=\; X_\tau \;\sim\; VG(C\tau, G, M)$$

Vanilla Option Pricing and Greeks

A closed form expression can be obtained for European Call and Put Options

$$e^{-r\tau}\mathbb{E}_{\mathbb{Q}}[S(T)-K|S(T)>K] = -S(t)e^{\omega\tau}\,\mathbf{c}\,(\mathbf{A}_P+\mathbf{I})^{-1}\,e^{(\mathbf{A}_P+\mathbf{I}_C)\,d}\,\mathbf{b}-K\,e^{-r\tau}\,(1-\Phi(d))$$

where

$$(1-\Phi(d)) = \int_d^{\infty}\mathbf{c}e^{\mathbf{A}_P x}\mathbf{b}dx \quad , \quad d = ln(K/S(t))-(r+w)(\tau) > 0$$

Analytic solutions also exist for the Greeks and for $d > 0$ we have

$$\frac{\partial C}{\partial S} = -e^{\omega\tau}\mathbf{c}(\mathbf{A}_P+\mathbf{I}_C)^{-1}e^{(\mathbf{A}_P+\mathbf{I}_C)d}\mathbf{b} + e^{\omega\tau}\mathbf{c}e^{(\mathbf{A}_P+\mathbf{I}_C)d}d - \frac{K}{S}e^{-r\tau}\mathbf{c}e^{\mathbf{A}_P d}\mathbf{b}$$

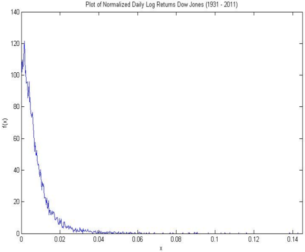
Value-at-Risk calculates a loss L for which a probability p exists of such a loss occurring

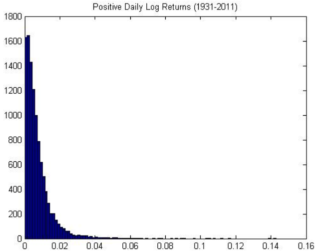
$$\begin{aligned} \mathbb{P}[S(T) < L] = \mathbb{P}[S(0)e^{((r+\omega)T+X_T)} < L] &= p \\ \mathbf{c}(\mathbf{A}_N)^{-1}e^{\mathbf{A}_N\,d_L}\mathbf{b} &= p \end{aligned}$$

where we solve for $d_L = ln\left(\frac{L}{S(0)}\right) - (r+\omega)T$

Approximating Empirical Data with EPT Densities using RARL2

Empirical Asset Returns





Problem: Derive the triple $(\tilde{\mathbf{A}}_P, \tilde{\mathbf{b}}_P, \tilde{\mathbf{c}}_P)$ such that

$$\left(log\big(\frac{S_{t+1}}{S_t}\big) \mid \frac{S_{t+1}}{S_t} \; > \; 1 \right) \sim EPT(\tilde{\mathbf{A}}_P, \tilde{\mathbf{b}}_P, \tilde{\mathbf{c}}_P)$$

where

$$\min_{\{\tilde{\mathbf{A}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}\}} ||\tilde{\mathbf{c}} e^{\tilde{\mathbf{A}}x} \tilde{\mathbf{b}} \; - \; \tilde{f}(x)||_2^2$$

Sampled Data

Transform in continuous time given by rational function

$$\begin{aligned}\tilde{F}(s) &= \int_0^\infty \tilde{\mathbf{c}}_P e^{\tilde{\mathbf{A}}_P x} \tilde{\mathbf{b}}_P e^{-sx} dx = \tilde{\mathbf{c}}_P (s\mathbf{I} - \tilde{\mathbf{A}}_P)^{-1} \tilde{\mathbf{b}}_P \\ &= \frac{1}{N} \sum_{k=0}^{N} e^{-sx_k}\end{aligned}$$

However RARL2 requires samples to be inputted in discrete time $F(z)$, $F(e^{i\theta})$. Isometry from Hanzon (1988) used to transform from continuous to discrete time

$$\tilde{F}(s) \mapsto F(z) = \frac{\sqrt{2}}{z-1} \tilde{F}\left(\frac{z+1}{z-1}\right) = \mathbf{c}_P (s\mathbf{I} - \mathbf{A}_P)^{-1} \mathbf{b}_P$$

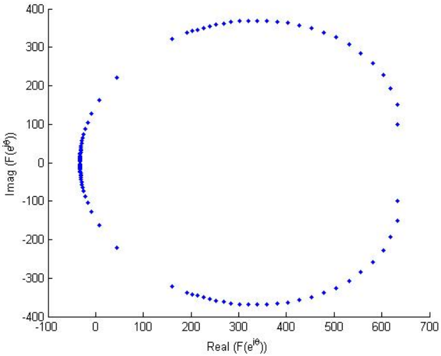
where the triple $(\mathbf{A}_P,\ \mathbf{b}_P,\ \mathbf{c}_P)$ is the discrete time realization.
The Mobius transformation was employed to achieve this

$$z \mapsto s = \frac{z+1}{z-1} \quad , \quad i\omega_j = \frac{e^{i\theta_j}+1}{e^{i\theta_j}-1}$$

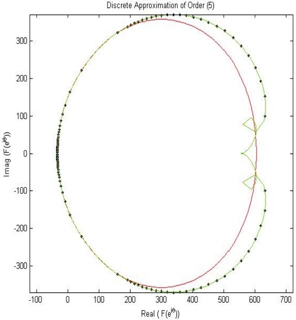
Sampled Data

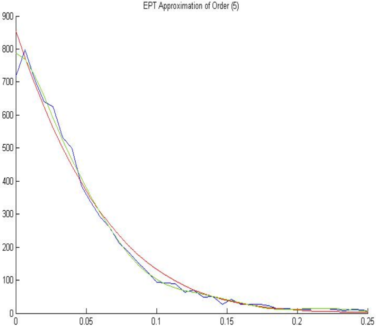
Hence we let $\theta \in [-\pi, \pi]$, solve for $i\omega$ and find

$$F(e^{i\theta_j}) = \frac{\sqrt{2}}{e^{i\theta_j} - 1} \left(\frac{1}{N} \sum_{k=0}^{N} e^{-i\omega_j x_k} \right)$$



Approximation with Sampled Data of Order 5





Non-Negative EPT Densities

Suppose $h(t)$ is the true function which we are approximating with the EPT density $f(t) = \mathbf{c}e^{\mathbf{A}t}\mathbf{b}$ the minimization problem is

$$\min_{\{\mathbf{A},\mathbf{b},\mathbf{c}|f(t)\geq 0\}} ||h(t) - f(t)||_2^2$$

Current RARL2 Algorithm: Minimize Eq. (5) w.r.t $\mathbf{A}, \mathbf{c}$ and then choose optimal $\mathbf{b}$

$$\min_{\{\mathbf{A},\mathbf{c}\}} ||h(t) - f(t)||_2^2 \qquad (5)$$

Perron-Frobenius type result: *suppose f(t) is non-negative* $\forall\ t \geq 0$ *then* $\lambda_M = max_{\lambda\in\sigma(A)}\mathbb{R}(\lambda)$ *is an element of the spectrum of A*

Hence a dominant real pole must be present in $\mathbf{A}$.

Non-Negative EPT Densities

Possible Solution: Try to identify the dominant pole and its coefficient, $\hat{\mu} e^{\hat{\lambda}_M t}$ from data or higher order system and exclude from approximation

$$\begin{aligned} \hat{h}(t) &= h(t) - \hat{\mu} e^{\hat{\lambda}_M t} \\ \hat{f}(t) &= f(t) - \hat{\mu} e^{\hat{\lambda}_M t} = \hat{\mathbf{c}} e^{\hat{\mathbf{A}} t} \hat{\mathbf{b}} \end{aligned}$$

$$\min_{\{\hat{\mathbf{A}}, \hat{\mathbf{c}}\}} ||\hat{h}(t) - \hat{f}(t)||_2^2 = \min_{\{\hat{\mathbf{A}}, \hat{\mathbf{c}}\}} ||\hat{h}(t) - \hat{\mathbf{c}} e^{\hat{\mathbf{A}} t} \hat{\mathbf{b}}||_2^2$$

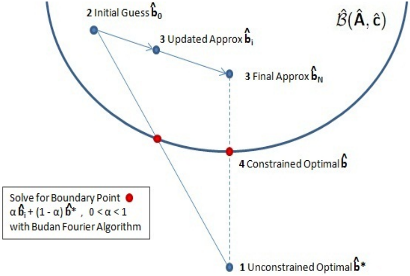
subject to $\mathbb{R}(\sigma(\hat{\mathbf{A}})) < \hat{\lambda}_M$

$\hat{\mathcal{B}}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ is a convex set for a given $(\hat{\mathbf{A}}, \hat{\mathbf{c}})$

Then there exists $\hat{\mathbf{b}} \in \hat{\mathcal{B}}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ such that

$$\hat{f}(t) = \hat{\mathbf{c}} e^{\hat{\mathbf{A}} t} \hat{\mathbf{b}} \geq -\hat{\mu} e^{\hat{\lambda}_M t} \quad \forall \ t \geq 0$$

Non-Negative EPT Densities



1. Find unconstrained optimal $\hat{\mathbf{b}}^*$
2. If $\hat{\mathbf{b}}^* \notin \hat{\mathcal{B}}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ then find initial estimate $\hat{\mathbf{b}}_0 \in \hat{\mathcal{B}}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$
3. Move from $\hat{\mathbf{b}}_i$ to $\hat{\mathbf{b}}_{i+1}$ based on L^2 distance of boundary point (in red) to $\hat{\mathbf{b}}^*$
4. Find constrained optimal $\hat{\mathbf{b}} \in \hat{\mathcal{B}}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ s.t. $\min_{\hat{\mathbf{b}}} ||\hat{\mathbf{b}} - \hat{\mathbf{b}}^*||_2^2$

Using Budan Fourier Algorithm Hanzon (2010), we can locate boundary point for $\hat{\mathbf{b}}_i$

It is a convex problem so we should be able to find constrained optimal

Conclusions and Further Research

Conclusions and Further Research

Results

- Define 2-EPT Densities on whole real line
- Demonstrate various calculations with 2-EPT Densities
- Illustrate benefits of using 2-EPT Approach
- Approximating EPT Densities with RARL2

Areas of Further Research with RARL2

- Approximate 2-EPT densities using RARL2 subject to the continuity condition $\mathbf{c}_N \mathbf{b}_N = \mathbf{c}_P \mathbf{b}_P$
- Apply RARL2 to forward rate curves including Svensson and Nelson-Siegal curves e.g. $y(t) = z_0 + z_1 e^{-\lambda_1 t} + z_2 t e^{-\lambda_2 t}$
- Constrain the approximating EPT functions to be non-negative

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Thank You

Questions