ERNSI 2012 Poster: EPT Levy Processes and Financial Applications

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Abstract

We consider a class of Levy processes that is characterised by the fact that at each point in time the process random variable has characteristic function that is a rational function or a real power of a rational function. A matrix calculus is available to deal with such processes. We consider how to deal with the requirement that the corresponding density functions have to be non-negative and we give conditions in terms of the poles and zeros of the characteristic function for it to be infinitely divisible (which implies that there is a corresponding Levy process). The Budan-Fourier type sequence of Exponential-Polynomial-Trigonometric functions plays an important role to verify non-negativity on a given finite interval. This can be applied to non-Gaussian option pricing and hedging problems. The 2-EPT Levy processes can be viewed as generalization of the variance-gamma Levy processes, which are well-known in financial mathematics. Further information and some software related to 2-EPT random variables and processes can be found on the website www.2-ept.com

Section 1: EPT and 2-EPT density functions

Consider real-valued EPT functions, of the form:

$$f(x) \;=\; Re(\sum_{k=1}^{K} p_k(x)\, e^{\mu_k x}) \qquad x \geq 0$$

$$f(x) \;=\; ce^{Ax}b \qquad \text{if } x \;\geq\; 0$$

A is an $(n \times n)$ matrix, c a $(1 \times n)$ row vector and b a $(n \times 1)$ column vector

- We can use (and if necessary reduce to) Minimal Realizations
- Class contains the real polynomials, real exponential functions and scaled real trigonometric polynomials
- Class of all EPT functions (union over all $n \in \mathbf{N}$) is closed under various operations
- Strictly proper Rational characteristic function and Laplace Transform

$$\phi(s) = \int_0^\infty e^{-sx} ce^{Ax} b dx = c(Is - A)^{-1} b = \frac{p(s)}{q(s)}$$

q is a polynomial of degree n while p is a polynomial of degree $m < n$, p and q coprime,
ϕ a rational function of McMillan degree n
Now we want to use these functions as Probability Density Functions $\rightarrow$ we need (A, b, c) to be such that

$$f(x) \;=\; ce^{Ax}b \;\geq 0 \qquad \forall\; x \;\geq\; 0$$

- Test for non-negativity, Hanzon and Holland (2010), (2012)
- Stability of A as eigenvalues located in open left plane

$$\lim_{x\to\infty} f(x) \;=\; \lim_{x\to\infty} ce^{Ax}b = 0$$

- Normalization due to finite integrability

A Perron-Frobenius type result for Non-Negative EPT Functions requires $\lambda_M \in \sigma(A)$ where

$$\lambda_M \;=\; \max_{\lambda\in\sigma(A)} Re(\lambda) \tag{1}$$

A 2-EPT density requires two EPT densities defined on either half real line

$$f(x) = \begin{cases} c_N\, e^{A_N\, x}\, b_N & \text{if } x < 0 \\ c_P\, e^{A_P\, x}\, b_P & \text{if } x > 0 \end{cases} \tag{2}$$

A_N is "anti-stable" with all eigenvalues located in open right half plane
Rational Fourier/Laplace Transform decomposed

$$\begin{aligned} \phi(s) \;&=\; \int_{-\infty}^{0} c_N\, e^{A_N\, x}\, b_N\, e^{-sx}\, dx \;+\; \int_0^\infty c_P\, e^{A_P\, x}\, b_P\, e^{-sx}\, dx \\ &=\; -c_N\, (Is \;-\; A_N)^{-1}\, b_N + c_P\, (Is \;-\; A_P)^{-1}\, b_P \\ &=\; \phi_N(s) \;+\; \phi_P(s) \;=\; \frac{p(s)}{q(s)} \end{aligned} \tag{3}$$

A natural generalisation of the EPT class also allows for a pointmass at zero

$$f(x) + d\delta, \qquad 0 \leq d \leq 1$$

where δ is the delta distribution and $f(x)$ a 2-EPT density function with mass $1 - d$
Proper Rational Fourier/Laplace Transform ("antistable+stable+constant")

$$\phi(s) \;=\; -c_N\, (Is \;-\; A_N)^{-1}\, b_N + c_P\, (Is \;-\; A_P)^{-1}\, b_P +\; d$$

EPT random variables and density functions allow for a large number of operations that can be expressed as formulas in terms of the corresponding (A, b, c) triples:

- Scaling
- Addition
- Multiplication
- Convolution
- Maximum and Minimum
- Composition
- Moments

Financial Mathematics Example: "Variance Gamma Density is 2-EPT"
Variance Gamma (C, G, M) Characteristic Function; Rational For C Integer

$$\phi(iu) \;=\; (\frac{G\, M}{G\, M + (M \;-\; G)iu + u^2})^C \tag{4}$$

$$f(x) = \begin{cases} c\, e^{A_N\, x}\, b & \text{if } x \leq 0 \\ c\, e^{A_P\, x}\, b & \text{if } x > 0 \end{cases}$$

$$b = (1, 0, 0, ..., 0)^T$$

$$c = (c_0, c_1, ..., c_{C-1}) \qquad , \qquad c_s = \frac{(MG)^C}{(C-1)!} \frac{(2(C-1) - s)!(G+M)^{-2C+1+s}}{(C-1-s)!}$$

$$A_N \;=\; \begin{pmatrix} M & 0 & \dots & 0 \\ -1 & M & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & -1 & M \end{pmatrix} \qquad , \qquad A_P \;=\; \begin{pmatrix} -G & 0 & \dots & 0 \\ 1 & -G & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 1 & -G \end{pmatrix}$$

Section 2: EPT and 2-EPT Levy processes

Lévy-Khintchine Formula *A probability law f of a real valued random variable is infinitely divisible with characteristic exponent Ψ,*

$$\int_{\mathbb{R}} e^{-isx} f(x) dx \;=\; e^{-\Psi(s)}, \qquad s \;\in\; \mathbb{R} \tag{5}$$

if and only if there exists a triple (a, σ, ν) where $a \in \mathbb{R}$, $\sigma \geq 0$ and ν a measure concentrated on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}} \min\{1, x^2\} d\nu(x) \;<\; \infty$, such that for all s in $\mathbb{R}$

$$\Psi(s) \;=\; ias + \frac{1}{2}\sigma^2 s^2 + \int_{\mathbb{R}} (1 - e^{isx} + isx\mathbb{I}_{\{|x|<1\}}) d\nu(x) \tag{6}$$

Infinite Divisibility of EPT Function: Consider

$$F(s) \;=\; c(sI - A)^{-1} b \tag{7}$$

If F is the Laplace Transform of an infinitely divisible distribution and $\Lambda(A, b, c) = \{s | c(sI - A)^{-1} b = 0\}$ then by Lukacs (1970)

$$\max_{\lambda\in\Lambda(A,b,c)} Re(\lambda) \leq \lambda_M \tag{8}$$

A triple (A, b, c) and F are called "minimum phase" if all its zeros and poles lie in the left half plane.
Feller (1971) states that a function F is the Laplace transform of an infinitely divisible probability distribution on $(0, \infty)$, if and only if $F = e^{-h}$ where h' is completely monotonic on $[0, \infty)$ and $h(0+) = 0$.

$$h'(s) \;=\; -\frac{F'(s)}{F(s)} \tag{9}$$

Widder (1941) gives a necessary and sufficient condition that a function h' is c.m. on $[0, \infty)$ if

$$h'(s) \;=\; \int_0^\infty e^{-sx} d\nu(x), \qquad s > 0, \tag{10}$$

where $\nu'(x)$ is a non-negative measure on $[0, \infty)$.
By Fubini

$$h(s) \;=\; \int_0^\infty \frac{1 - e^{-sx}}{x} d\nu(x), \qquad s > 0. \tag{11}$$

It can be seen that

$$h'(s) = -\frac{F'(s)}{F(s)} \;=\; \int_0^\infty e^{-sx} \Big(Tr(e^{Ax}) - Tr(e^{Bx})\Big) dx. \tag{12}$$

where B is the companion matrix of the polynomial $c(sI - A)^* b$, where $(sI - A)^* = (sI - A)^{-1}|sI - A|$ denotes the adjoint of $sI - A$..

Theorem *Given a minimal, minimum phase triple (A, b, c) that defines a not identically zero EPT probability density function f. Let B be the companion matrix of $c(sI - A)^* b$ and $F(s)$ is the Laplace transform of $f(x) = ce^{Ax}b$, then*

$$F(s) \;=\; \exp\Big(-\int_0^\infty (1 - e^{-sx}) \frac{Tr(e^{Ax}) - Tr(e^{Bx})}{x} dx\Big), \tag{13}$$

Thus f is infinitely divisible iff $Tr(e^{Ax}) \geq Tr(e^{Bx})$ for all $x \geq 0$
Lévy Measure has density

$$\nu'(x) \;=\; \frac{Tr(e^{Ax}) - Tr(e^{Bx})}{x} \tag{14}$$

EPT Levy process is of Finite Variation as $\sigma = 0$ and

$$a \;=\; -\int_0^1 \Big(Tr(e^{Ax}) - Tr(e^{Bx})\Big) dx \;>\; -\infty \tag{15}$$

Similar result holds for mixtures of EPT Functions with Pointmass
Infinitely Divisible 2-EPT Functions.
Let F be the rational Laplace Transform of the 2-EPT Probability Density Function f with realization $(A_N, b_N, c_N, A_P, b_P, c_P, d)$

$$F(s) \;=\; -c_N(sI - A_N)^{-1} b_N + c_P(sI - A_P)^{-1} b_P + d \tag{16}$$

F can be factored into two rational functions F_1 and F_2 where

$$F(s) \;=\; F_1(s)\; F_2(s) \tag{17}$$

The factors F_1 and F_2 are rational functions with all their poles and zeros located in the open left and open right half planes respectively.
The 2-EPT Probability Density Function f is infinitely divisible iff F_1 and F_2 satisfy the "one sided" EPT infinite divisibility conditions.
Note that the infinite divisibility condition has a very elegant formulation in terms of the poles $\lambda_1, \dots, \lambda_n$ and zeros $\nu_1, \dots, \nu_m$ of the rational transform of an EPT function, namely that

$$\sum_{k=1}^{n} e^{\lambda_k x} - \sum_{j=1}^{m} e^{\nu_j} \geq 0 \text{ for all } x \geq 0$$

For a 2-EPT function with transform F this condition has to hold for the ("Wiener-Hopf") factors F_1 and F_2 with the understanding that for F_2, the "anti-stable, anti-minimum phase" factor, the condition has to hold on the negative half line $x \leq 0$.

Section 3: 2-EPT Option Pricing

Risk Neutral 2-EPT Price Process
Price process under the real world measure $\mathbb{P}$

$$S(T) \;=\; S(t)e^{X_\tau} \tag{18}$$

where $\tau = (T - t)$, $X_\tau \sim\; 2 - EPT(A_N, b_N, c_N, A_P, b_P, c_P)$ and the probability density function for X_τ is infinitely divisible
Price process under the risk neutral measure $\mathbb{Q}$

$$S(T) \;=\; S(t)e^{(r+w)\tau + X_\tau} \tag{19}$$

w is chosen such that

$$\mathbb{E}_{\mathbb{Q}}[S(T)] \;=\; S(t)e^{r\tau} \tag{20}$$

Discounted asset price is a martingale and w_T calculated explicitly

$$w_T \;=\; -\log(c_N(I + A_N)^{-1} b_N - c_P(I + A_P)^{-1} b_P) \tag{21}$$

European Call Option Prices

$$C(S, \tau, K) = e^{-r\tau}\; \mathbb{E}_{\mathbb{Q}}[S(T) - K]^+ \tag{22}$$

The price for the European call option as outlined above for $d \;>\; 0$ is given by

$$\begin{aligned} C(S, \tau, K) \;&=\; S(t)e^{\omega\tau}\Big(c_N(A_N + I)^{-1} b_P - c_N(A_N + I)^{-1} e^{-(A_N+I)d} b_N - c_P(A_P + I)^{-1} b_P\Big) \\ &-\; K\, e^{-r\tau}\, (1 - c_N A_N^{-1} e^{-A_N d} b_N) \end{aligned}$$

A similar expression for the price can be obtained when $d \;\leq\; 0$.

$$\begin{aligned} C(S, \tau, K) \;&=\; -S(t)e^{\omega\tau} c_P(A_P + I)^{-1} e^{-(A_P+I)d} b_P \\ &+\; K\, e^{-r\tau}\; c_P A_P^{-1} e^{-A_P d} b_P \end{aligned}$$

where $d = \log(S(t)/K) + (r + \omega)\tau$
Remark: The Delta and Gamma of the European Call can be calculated explicitly in a similar fashion.
Discretely Monitored Lookback Option
Let the discretely monitored risk neutral price process be

$$S(T) = S(t)e^{\sum_{i=0}^{N} Y_i} \qquad , \qquad \mathbb{E}_{\mathbb{Q}}[e^{Y_i}] = e^{r\delta t} \tag{23}$$

where $Y_i \sim\; 2 - EPT(A_{N_i}, b_{N_i}, c_{N_i}, A_{P_i}, b_{P_i}, c_{P_i})$ for all $i = \{1, 2, ..., N\}$ and $Y_0 = 0$.
The minimum of the Discrete Time 2-EPT process

$$M(N) \;=\; \min_{n\in\{0,1,2,...,N\}} \sum_{i=0}^{n} Y_i \tag{24}$$

has a generalised EPT distribution which can be computed via

$$M(T) \;=\; \min\{0\;,\; X_1 \;+\; \min\{0,\; X_2 \;+\; \min\{0,\; X_3 \;+\; \dots\}\}\} \tag{25}$$

An equality in distribution ($\overset{d}{=}$) for $M(T + 1)$ in terms of $M(T)$ is defined as

$$M(T + 1) \;\overset{d}{=}\; \min\{0,\; X_{T+1} \;+\; M(T)\} \tag{26}$$

Discretely Monitored Lookback Options with Fixed Strike:
$M(T)$:=Density of Minimum of Discrete Time 2-EPT Process of Length T

$$M(T) \sim EPT(A_M, b_M, c_M, d_M)$$

Lookback Option with Fixed Strike T

$$L(S(T), T; T, K) \;=\; \max\{K - S_{\min}(T), 0\} \tag{27}$$

Risk Neutral Option Price at time 0 with $d \leq 0$

$$L(S(0), 0; T, K) \;=\; Ke^{-rT} c_M A_M^{-1} e^{A_M d} b_M \;-\; S(0)e^{-rT} c_M(A_M + I)^{-1} e^{(A_M+I)d} b_M$$

where $d = \log(K/S(0))$

References

Feller, W. *An to Probability Theory and its Applications, Vol. II* John Wiley and sons, 1971

Hanzon, B., Holland, F. *"Non-Negativity of Exponential Polynomial Trigonometric Functions - A Budan Fourier Sequence Approach"* BFS Toronto Poster 429, June 22 - 26, 2010. Available at "www.2-ept.com"

Hanzon, B., Holland, F. *Non-negativity analysis for Exponential-Polynomial-Trigonometric Functions on the non-negative real half-line* in: Wolfgang Arendt (Editor), Joseph A. Ball (Editor), Jussi Behrndt (Editor), Karl-Heinz Foerster (Editor), Volker Mehrmann (Editor), Carsten Trunk (Editor) *Spectral Theory, Mathematical System Theory, Evolution Equations, Differential and Difference Equations*, Birkhaeuser, Basel, 2012.

Hanzon, B., Olivi, M., Sexton, C., *Rational Approximation of Transfer Functions for Non-Negative EPT Densities* Conference Proceedings, SYSID 2012

Lukacs, E.,*Characteristic Functions 2nd Edition*, Griffen, London, 1970

Widder, D.V. *The Laplace Transforms* Princeton University Press, 1941

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