Convex Optimisation for Non-Negative EPT Functions with Budan-Fourier Algorithm in Matlab

We describe the implementation of the convex optimisation procedure illustrated in Section (5) of Sexton, Olivi, Hanzon (2011).
Using RARL2 or otherwise we wish the minimize the criterion in Eq. (1) over the minimal triple $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ of order n where $\mathbf{A}$ is stable such that $\sigma(\mathbf{A}) < 0$.

$$\min_{\{\mathbf{A},\mathbf{b},\mathbf{c}\}} ||h(x) - \mathbf{c}e^{\mathbf{A}x}\mathbf{b}||_2^2 \qquad , \qquad \forall\, x \;\geq\; 0 \tag{1}$$

which is satisfied by the optimal triple $(\mathbf{A}^*, \mathbf{b}^*, \mathbf{c}^*)$. We are seeking to approximate $h(x)$ with a non-negative EPT function implying the criterion we wish to minimize is

$$\min_{\{\mathbf{A},\mathbf{b},\mathbf{c}|\mathbf{c}e^{\mathbf{A}x}\mathbf{b}\geq 0\}} ||h(x) - \mathbf{c}e^{\mathbf{A}x}\mathbf{b}||_2^2 \qquad , \qquad \forall\, x \;\geq\; 0 \tag{2}$$

To ensure non-negativity it is necessary that the spectrum of $\mathbf{A}$ contains a dominant real pole as was stated in Hanzon and Holland (2010b). From Sexton, Olivi, Hanzon (2011) we estimate the dominant real pole, λ, and is coefficient μ. These parameters may be approximated from the data or a higher order system but subject the condition that $\mu > 0$ and $\lambda < 0$. Hence we define a new triple of order $n - 1$ such that

$$\mathbf{c}e^{\mathbf{A}x}\mathbf{b} \;=\; \hat{\mathbf{c}}e^{\hat{\mathbf{A}}x}\hat{\mathbf{b}} + \mu e^{\lambda}$$

The minimization problem is now given by

$$\min_{\{\hat{\mathbf{A}},\hat{\mathbf{b}},\hat{\mathbf{c}}|\hat{\mathbf{c}}e^{\hat{\mathbf{A}}x}\hat{\mathbf{b}}\geq -\mu e^{\lambda x}\}} ||(h(x) - \mu e^{\lambda x}) - \hat{\mathbf{c}}e^{\hat{\mathbf{A}}x}\hat{\mathbf{b}}||_2^2 \qquad , \qquad \forall\, x \;\geq\; 0 \tag{3}$$

subject to $\sigma(\hat{\mathbf{A}}) < \lambda$ which is a relatively simply constraint to impose. In fact we transform this constraint to a stability constraint by weighting the criterion function as

$$\min_{\{\hat{\mathbf{A}},\hat{\mathbf{b}},\hat{\mathbf{c}}|\hat{\mathbf{c}}e^{\tilde{\mathbf{A}}x}\hat{\mathbf{b}}\geq -\mu\}} ||(h(x)e^{-\lambda x} - \mu) - \hat{\mathbf{c}}e^{\tilde{\mathbf{A}}x}\hat{\mathbf{b}}||_2^2 \qquad , \qquad \forall\, x \;\geq\; 0 \tag{4}$$

where $\tilde{\mathbf{A}} \;=\; \hat{\mathbf{A}} - \lambda\mathbf{I}$ and $\sigma(\tilde{\mathbf{A}}) < 0$

We now have the criterion function given by Eq. (4). However RARL2 minimizes the criterion over the pair $(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ and then solves for an optimal $\hat{\mathbf{b}}$. Hence the RARL2 criterion function is

$$\min_{\{\hat{\mathbf{A}},\hat{\mathbf{c}}\}} ||(h(x)e^{-\lambda x} - \mu) - \hat{\mathbf{c}}e^{\tilde{\mathbf{A}}x}\hat{\mathbf{b}}||_2^2 \qquad , \qquad \forall\, x \;\geq\; 0 \tag{5}$$

We will denote the optimal triple satisfying Eq. (5) as $(\hat{\mathbf{A}}^*, \hat{\mathbf{b}}^*, \hat{\mathbf{c}}^*)$. We define the convex set $\mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)$ such that if $\hat{\mathbf{b}}^* \;\in\; \mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)$ then we have

$$\hat{\mathbf{c}}^* e^{\hat{\mathbf{A}}^* x}\hat{\mathbf{b}}^* \;>\; -\mu$$

which implies non-negativity in the original density given by

$$\hat{\mathbf{c}}^* e^{(\hat{\mathbf{A}}^* + \lambda_M)x}\hat{\mathbf{b}}^* + \mu e^{\lambda_M x} \;\geq\; 0$$

However if $\hat{\mathbf{b}}^* \notin \mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)$ then we must find $\hat{\mathbf{b}} \in \mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)$ which can be found by minimizing the criterion

$$\min_{\{\hat{\mathbf{b}} \in \mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)\}} ||\hat{\mathbf{c}}^* e^{\hat{\mathbf{A}}^* x} \hat{\mathbf{b}}^* - \hat{\mathbf{c}}^* e^{\hat{\mathbf{A}}^* x} \hat{\mathbf{b}}||_2^2 = \min_{\{\hat{\mathbf{b}}_C^* \in \mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)\}} ||\hat{\mathbf{c}}^* e^{\hat{\mathbf{A}}^* x} \left(\hat{\mathbf{b}}^* - \hat{\mathbf{b}}\right)||_2^2 \tag{6}$$

Since $\mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)$ is a convex set we should be to locate $\hat{\mathbf{b}}$ which minimises Eq. (6). If $\hat{\mathbf{b}}^* \notin \mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)$ then global optimal $\hat{\mathbf{b}}$ satisfying Eq. (6) must lie on the boundary of the set. Any vector which lies on the boundary of $\mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)$ is denoted $\hat{\mathbf{B}}$. Hence the solution of Eq. (6) is denoted $\hat{\mathbf{B}}^*$.

If we define the observability grammian as the solution to the Lyapunov equation

$$(\hat{\mathbf{A}}^*)^T \mathbf{Q} + \mathbf{Q}(\hat{\mathbf{A}}^*) + (\hat{\mathbf{c}}^*)^T \hat{\mathbf{c}}^* = 0$$

and for notation purposes we set

$$||\hat{\mathbf{b}}||_{\mathbf{Q}}^2 = \hat{\mathbf{b}}^T \mathbf{Q} \hat{\mathbf{b}} = ||\hat{\mathbf{c}}^* e^{\hat{\mathbf{A}}^* x} \hat{\mathbf{b}}||_2^2 \quad , \quad \forall\, x \geq 0 \tag{7}$$

We can then represent Eq. (6) as

$$\min_{\{\hat{\mathbf{b}} \in \mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)\}} ||\hat{\mathbf{b}}^* - \hat{\mathbf{b}}||_{\mathbf{Q}}^2 \tag{8}$$

As we have restricted $\mu > 0$ we know that $0 \in \mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)$. Hence we can see that we must have

$$||\hat{\mathbf{b}}||_{\mathbf{Q}}^2 \leq ||\hat{\mathbf{b}}^*||_{\mathbf{Q}}^2 \tag{9}$$

for any $\hat{\mathbf{b}}$ we consider in our in convex set $\mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)$. Owing to this constraint and the presence of the dominant real pole it is shown in Section 6 of Sexton, Olivi, Hanzon (2011) how to construct $T > 0$ such that if

$$\hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} \hat{\mathbf{b}} + \mu e^{\lambda_M x} > 0 \quad , \quad \forall\, x \in [0, T]$$

then non-negativity holds for all $x \geq 0$.

For the convex optimisation procedure we denote the starting point as $\hat{\mathbf{b}}_0$. A reasonable initial guess may be the zero vector so we set $\hat{\mathbf{b}}_0 = 0$. Using the Budan-Fourier algorithm it is possible to derive $\alpha \in [0, 1]$ such that the convex combination

$$\hat{\mathbf{B}}_0 = \alpha \hat{\mathbf{b}}_0 + (1 - \alpha) \hat{\mathbf{b}}^* \tag{10}$$

lies on the boundary of $\mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)$. We then calculate the norm $||\hat{\mathbf{B}}_0 - \hat{\mathbf{b}}^*||_{\mathbf{Q}}^2$. For step i and given $\hat{\mathbf{b}}_i$ we use an efficient gradient search algorithm to find $\hat{\mathbf{b}}_{i+1}$. We then derive $\hat{\mathbf{B}}_{i+1}$

$$\hat{\mathbf{B}}_{i+1} = \alpha \hat{\mathbf{b}}_{i+1} + (1 - \alpha) \hat{\mathbf{b}}^* \tag{11}$$

which minimizes the norm

$$||\hat{\mathbf{B}}_{i+1} - \hat{\mathbf{b}}^*||_{\mathbf{Q}}^2 \tag{12}$$

We continue in this manner until the norm in Eq. (12) can no longer be improved upon.

The convex optimisation algorithm must always ensure $\hat{\mathbf{b}}_i \in \mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)$ which can be checked using the Budan-Fourier method. It is for this reason that we choose to using the convex optimisation algorithm by searching the set $\mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)$ with $\hat{\mathbf{b}}$ rather than the vector on the boundary $\hat{\mathbf{B}}$.

The optimisation procedure must also ensure the condition in Eq. (9) on the $\mathbf{Q}$ norm of $\hat{\mathbf{b}}$ is also enforced.

EPT_ConvexOP_MinSearch.m

Accepts as inputs the optimal triple $(\hat{\mathbf{A}}^*,\ \hat{\mathbf{b}}^*,\ \hat{\mathbf{c}}^*)$ of McMillan degree $(n-1)$, the dominant pole and coefficient μ with λ and the initial guess of $\hat{\mathbf{b}}_0$. The code first transforms the $\hat{\mathbf{A}}^*$ matrix such that $\tilde{\mathbf{A}}^* = \hat{\mathbf{A}}^* - \lambda\mathbf{I}$. The `lyap` command is then used to calculate the observability grammian $\mathbf{Q}$ for the pair $(\hat{\mathbf{A}}^*,\ \hat{\mathbf{c}}^*)$. The m-file `Finite_T.m` as described below calculates the finite T such that examining non-negativity on $[0,\ T]$ is sufficient to checking for sign-changing zeros on the half real line $[0, \infty)$.
The convex optimisation procedure is then conducted using the inbuilt Matlab search algorithm `fminsearch` where `Convex_Optimisation.m` calculates the quantity to be minimised and returns the final $\hat{\mathbf{b}}$. Using `FindingAlpha.m` we calculate the corresponding $\hat{\mathbf{B}}^*$ which is a linear combination of $\hat{\mathbf{b}}$ and $\hat{\mathbf{b}}^*$. A plot of the resulting EPT functions is then given.

FindingAlpha.m

Returns the vector $\hat{\mathbf{B}}$ which is a linear combination of $\hat{\mathbf{b}}^*$ and $\hat{\mathbf{b}}$ as shown in Eq (11) such that $\hat{\mathbf{B}}$ lies on the boundary of $\mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)$. A standard bisection calculates the convex combination using the Budan-Fourier algorithm to test if the combination of EPT functions has sign-changing zeros on $[0, T]$.

Convex_Optimisation.m

Calculates $\hat{\mathbf{B}}_i$ on the boundary of $\mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)$ for a given $\hat{\mathbf{b}}_i$ and $\hat{\mathbf{b}}^*$. If $\hat{\mathbf{b}}_i \notin \mathcal{B}(\hat{\mathbf{A}}^*, \hat{\mathbf{c}}^*)$ or $||\hat{\mathbf{b}}_i||^2_{\mathbf{Q}} > ||\hat{\mathbf{b}}^*||^2_{\mathbf{Q}}$ there is an immediate penalty and the search algorithm readjusts to $\hat{\mathbf{b}}_{i-1}$. Otherwise the norm $||\hat{\mathbf{b}}_i - \hat{\mathbf{b}}^*||^2_{\mathbf{Q}}$ is calculated.

Finite_T.m

The inputs required are the optimal minimal triple $(\hat{\mathbf{A}}^*,\ \hat{\mathbf{b}}^*,\ \hat{\mathbf{c}}^*)$ of the EPT function with its dominant pole and coefficient, μ and λ. Following the logic in Section 6 of Sexton, Olivi, Hanzon (2011) we derive $\lambda_{\min},\ \tilde{\lambda}_{\min}$ and R^2 yielding $\epsilon = \mu\lambda_{\min}\tilde{\lambda}_{\min}/R^2$. Using a dummy variable we construct a large T_1 such that $V(T_1) < \epsilon$. Then using bisection we calculate T such that $V(T_1) < \epsilon$.

Zeros_EPT_DP.m

The m-file accepts as inputs an EPT function defined using the triple $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ and dominant pole, λ, with coefficient μ. A $T > 0$ is also required an input. The m-file then uses the Budan-Fourier algorithm as described in "http://www.2-ept.com/budan-fourier-test-for-non-negativity-of-ept-functions.html" to test whether the EPT function with dominant real pole has sign-changing zeros on the finite interval $[0, T]$. This file very similar to `Zeros_EPT.m` from the Budan-Fourier algorithm except it simply returns an indicator, 0 or 1 implying whether sign-changing zeros are present or not.

Example

```
mu = 16;
lambda = -0.5;
c = [-30 15];
A = [-1 0;0 -2];
b_star = [1;1];
b_0 = [0.950;2.175];
EPT_ConvexOP_MinSearch(A,c,b_0,b_star,lambda,mu)
```

References

Sexton, C., Olivi, M., Hanzon, B. *Rational Approximation of Transfer Functions for Non-Negative EPT Densities* SYSID Conference,

Hanzon, B., Holland, F. *On a Perron Frobenius type result for Non-Negative Impulse Response Functions* "http://euclid.ucc.ie/pages/staff/hanzon/HanzonHollandDominantPoleLisbonPaperisn.pdf"