Financial Modelling with 2-EPT Lévy Processes

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INTRODUCTION

EPT Functions on Half Line

$$f(x) = Re\Big(\sum_{k=1}^{K} p_k(x)\, e^{\mu_k x}\Big) \qquad x \geq 0$$

$$f(x) = \mathbf{c}e^{\mathbf{A}x}\mathbf{b} \qquad \text{if } x \geq 0$$

$\mathbf{A}$ is an $(n \times n)$ matrix, $\mathbf{c}$ a $(1 \times n)$ row vector and $\mathbf{b}$ a $(n \times 1)$ column vector

- Class contains the real polynomials, real exponential functions and scaled real trigonometric polynomials
- Closed under various operations
- Strictly proper rational characteristic function and Laplace Transform

$$\phi(s) = \int_0^\infty e^{-sx}\mathbf{c}e^{\mathbf{A}x}\mathbf{b}dx = \mathbf{c}(\mathbf{I}s - \mathbf{A})^{-1}\mathbf{b} = \frac{p(s)}{q(s)}$$

q is a polynomial of degree n while p is a polynomial of degree $m < n$

ϕ a rational function of McMillan degree n

EPT Density

Triple $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ such that

$$f(x) = \mathbf{c}e^{\mathbf{A}x}\mathbf{b} \geq 0 \qquad \forall\, x \geq 0$$

- Test for non-negativity, Hanzon and Holland (2010)
- Minimal Realizations
- Stability of $\mathbf{A}$ as eigenvalues located in open left plane

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \mathbf{c}e^{\mathbf{A}x}\mathbf{b} = 0$$

- Normalization due to finite integrability

A Perron-Frobenius result for Non-Negative EPT Functions requires $\lambda_M \in \sigma(\mathbf{A})$

$$\max_{\lambda\in\sigma(\mathbf{A})} Re(\lambda) = \lambda_M \qquad (1)$$

2-EPT Density

A 2-EPT density requires two EPT densities defined on either half real line

$$f(x) = \begin{cases} \mathbf{c}_N \; e^{\mathbf{A}_N \; x} \; \mathbf{b}_N & \text{if } \; x \leq 0 \\ \\ \mathbf{c}_P \; e^{\mathbf{A}_P \; x} \; \mathbf{b}_P & \text{if } \; x > 0 \end{cases} \tag{2}$$

$\mathbf{A}_N$ is "anti-stable" with all eigenvalues located in open right half plane
Rational Laplace Transform decomposed

$$\begin{aligned} \phi(s) &= \int_{-\infty}^{0} C_N \; e^{\mathbf{A}_N \; x} \; \mathbf{b}_N \; e^{-sx} \; dx \; + \; \int_{0}^{\infty} \mathbf{c}_P \; e^{\mathbf{A}_P \; x} \; \mathbf{b}_P \; e^{-sx} \; dx \\ &= -\mathbf{c}_N \; (\mathbf{I}s \; - \; \mathbf{A}_N)^{-1} \; \mathbf{b}_N + \mathbf{c}_P \; (\mathbf{I}s \; - \; \mathbf{A}_P)^{-1} \; \mathbf{b}_P \\ &= \phi_N(s) \; + \; \phi_P(s) \; = \frac{p(s)}{q(s)} \end{aligned} \tag{3}$$

Point Mass at Zero

A natural generalisation of the EPT class also allows for a pointmass at zero

$$f(x) + \mathbf{d}\delta, \qquad 0 \leq d \leq 1$$

where δ is the delta distribution and $f(x)$ a 2-EPT density function

Proper rational Laplace Transform

$$\phi(s) \;=\; -\mathbf{c}_N \left(\mathbf{I}s \;-\; \mathbf{A}_N\right)^{-1} \mathbf{b}_N + \mathbf{c}_P \left(\mathbf{I}s \;-\; \mathbf{A}_P\right)^{-1} \mathbf{b}_P + \;\mathbf{d}$$

2-EPT / EPT Functions Closed Under Operations

- Scaling
- Addition
- Multiplication
- Convolution
- Maximum and Minimum
- Composition
- Moments

Variance Gamma Density is 2-EPT

Variance Gamma (C, G, M) Characteristic Function; Rational For C Integer

$$\phi(is) = \left(\frac{G\ M}{G\ M + (M - G)is + u^2}\right)^C \qquad (4)$$

$$f(x) = \begin{cases} \mathbf{c}\ e^{\mathbf{A}_N\ x}\ \mathbf{b} & \text{if } x \leq 0 \\ \mathbf{c}\ e^{\mathbf{A}_P\ x}\ \mathbf{b} & \text{if } x > 0 \end{cases}$$

$$\mathbf{b} = (1, 0, 0, ..., 0)^T$$

$$\mathbf{c} = (c_0, c_1, ..., c_{C-1}) \quad , \quad c_s = \frac{(MG)^C}{(C-1)!} \frac{(2(C-1)-s)!(G+M)^{-2C+1+s}}{(C-1-s)!}$$

$$\mathbf{A}_N = \begin{pmatrix} M & 0 & \dots & 0 \\ -1 & M & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & -1 & M \end{pmatrix} \quad , \quad \mathbf{A}_P = \begin{pmatrix} -G & 0 & \dots & 0 \\ 1 & -G & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 1 & -G \end{pmatrix}$$

2-EPT Lévy Processes

Lévy Khintchine Formula

Lévy-Khintchine Formula *A probability law f of a real valued random variable is infinitely divisible with characteristic exponent Ψ,*

$$\int_{\mathbb{R}} e^{-isx} f(x)dx \; = \; e^{-\Psi(s)}, \qquad s \; \in \; \mathbb{R} \tag{5}$$

if and only if there exists a triple (a, σ, ν) where $a \in \mathbb{R}$, $\sigma \geq 0$ and ν a measure concentrated on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}} \min\{1, x^2\} d\nu(x) \; < \; \infty$, such that

$$\Psi(s) \; = \; ias + \frac{1}{2}\sigma^2 s^2 + \int_{\mathbb{R}} (1 - e^{isx} + isx\mathbb{I}_{\{|x|<1\}})d\nu(x) \tag{6}$$

for all s in $\mathbb{R}$.

Infinite Divisibility of EPT Function

$$F(s) \; = \; \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} \tag{7}$$

If F is the Laplace Transform of an infinitely divisible distribution and $\Lambda(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \{s | \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = 0\}$ then by Lukacs (1970)

$$\max_{\lambda \in \Lambda(\mathbf{A},\mathbf{b},\mathbf{c})} Re(\lambda) \leq \lambda_M \tag{8}$$

A rational transfer function is known as "minimum phase" if all its zeros and poles are located in the same half plane.

Feller (1971) states that a function F is the Laplace transform of an infinitely divisible probability distribution on $(0, \infty)$, if and only if $F = e^{-h}$ where h' is completely monotonic on $[0, \infty)$ and $h(0+) = 0$.

$$h'(s) \; = \; -\frac{F'(s)}{F(s)} \tag{9}$$

Infinite Divisibility of EPT Function

Widder (1941) gives a necessary and sufficient condition that a function h' is c.m. on $[0, \infty)$ if

$$h'(s) = \int_0^\infty e^{-sx} d\nu(x), \qquad s > 0, \tag{10}$$

where $\nu'(x)$ is a non-negative measure on $[0, \infty)$.
By Fubini

$$h(s) = \int_0^\infty \frac{1 - e^{-sx}}{x} d\nu(x), \qquad s > 0. \tag{11}$$

It can be seen that

$$h'(s) = -\frac{F'(s)}{F(s)} = \int_0^\infty e^{-sx} \Big(Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}x}) \Big) dx. \tag{12}$$

where $\mathbf{B}$ is the companion matrix of $\mathbf{c}(s\mathbf{I} - \mathbf{A})^* \mathbf{b}$.

EPT Lévy Triple (a, σ, ν)

Theorem *Given a minimal triple* $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ *that defines a not identically zero EPT probability density function* f *of minimum phase. Let* $\mathbf{B}$ *be the companion matrix of* $\mathbf{c}(s\mathbf{I} - \mathbf{A})^* \mathbf{b}$ *and* $F(s)$ *is the Laplace transform of* $f(x) = \mathbf{c}e^{\mathbf{A}x}\mathbf{b}$*, then*

$$F(s) = \exp\left(-\int_0^\infty (1 - e^{-sx}) \frac{Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}x})}{x} dx \right), \tag{13}$$

Thus f *is infinitely divisible if* $Tr(e^{\mathbf{A}x}) \geq Tr(e^{\mathbf{B}x})$ *for all* $x \geq 0$

Lévy Measure

$$\nu'(x) = \frac{Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}x})}{x} \tag{14}$$

EPT Levy process is of Finite Variation as $\sigma = 0$ and

$$a = -\int_0^1 \left(Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}x}) \right) dx < \infty \tag{15}$$

Similar result holds for mixtures of EPT Functions with Pointmass

Infinitely Divisible 2-EPT Functions

Let F be the rational Laplace Transform of the 2-EPT Probability Density Function f with realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P, \mathbf{d})$

$$F(s) = -\mathbf{c}_N(s\mathbf{I} - \mathbf{A}_N)^{-1}\mathbf{b}_N + \mathbf{c}_P(s\mathbf{I} - \mathbf{A}_P)^{-1}\mathbf{b}_P + \mathbf{d} \qquad (16)$$

F can be factored into two rational functions F_1 and F_2 where

$$F(s) = F_1(s)\, F_2(s) \qquad (17)$$

The factors F_1 and F_2 are rational functions with all their poles and zeros located in the open left and open right half planes respectively.

The 2-EPT Probability Density Function f is infinitely divisible if F_1 and F_2 are infinitely divisible.

2-EPT Option Prices

Risk Neutral 2-EPT Price Process

Price process under the real world measure $\mathbb{P}$

$$S(T) \;=\; S(t)e^{X_\tau} \tag{18}$$

where $\tau = (T - t)$, $X_\tau \sim\ 2 - EPT(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$ and the probability density function for X_τ is infinitely divisible

Price process under the risk neutral measure $\mathbb{Q}$

$$S(T) \;=\; S(t)e^{(r+\omega)\tau + X_\tau} \tag{19}$$

ω is chosen such that

$$\mathbb{E}_{\mathbb{Q}}[S(T)] \;=\; S(t)e^{r\tau} \tag{20}$$

Discounted asset price is a martingale and $\omega\tau$ calculated explicitly

$$\omega\tau \;=\; -\log(\mathbf{c}_N(\mathbf{I} + \mathbf{A}_N)^{-1}\mathbf{b}_N - \mathbf{c}_P(\mathbf{I} + \mathbf{A}_P)^{-1}\mathbf{b}_P) \tag{21}$$

European Call Option Prices

$$C(S,\tau,K) = e^{-r\tau}\, \mathbb{E}_{\mathbb{Q}}[S(T) - K]^{+} \tag{22}$$

The price for the European call option as outlined above for $d > 0$ is given by

$$\begin{aligned} C(S,\tau,K) &= S(t)e^{\omega\tau}\Big(\mathbf{c}_N(\mathbf{A}_N + \mathbf{I})^{-1}\mathbf{b}_P - \mathbf{c}_N(\mathbf{A}_N + \mathbf{I})^{-1}e^{-(\mathbf{A}_N+\mathbf{I})d}\mathbf{b}_N - \mathbf{c}_P(\mathbf{A}_P + \mathbf{I})^{-1}\mathbf{b}_P\Big) \\ &- K\, e^{-r\tau}\, (1 - \mathbf{c}_N\mathbf{A}_N^{-1}e^{-\mathbf{A}_N d}\mathbf{b}_N) \end{aligned}$$

A similar expression for the price can be obtained when $d \leq 0$.

$$\begin{aligned} C(S,\tau,K) &= -S(t)e^{\omega\tau}\mathbf{c}_P(\mathbf{A}_P + \mathbf{I})^{-1}e^{-(\mathbf{A}_P+\mathbf{I})d}\mathbf{b}_P \\ &+ K\, e^{-r\tau}\, \mathbf{c}_P\mathbf{A}_P^{-1}e^{-\mathbf{A}_P d}\mathbf{b}_P \end{aligned}$$

where $d = \log(S(t)/K) + (r + \omega)\tau$

Delta and Gamma

For a European Call Option with $d \leq 0$, Delta is

$$\begin{aligned}\frac{\partial C(S,\tau,r,K)}{\partial S} &= -e^{\omega\tau}\mathbf{c}_P(\mathbf{A}_P+\mathbf{I})^{-1}e^{-(\mathbf{A}_P+\mathbf{I})d}\mathbf{b}_P + e^{\omega\tau}\mathbf{c}_P e^{-(\mathbf{A}_P+\mathbf{I})d}\mathbf{b}_P \\ &- \frac{K}{S}\,e^{-r\tau}\,\mathbf{c}_P e^{-\mathbf{A}_P d}\mathbf{b}_P\end{aligned}$$

For $d \leq 0$ Gamma is given by

$$\begin{aligned}\frac{\partial^2 C(S,\tau,r,K)}{\partial S^2} &= \frac{e^{\omega\tau}}{S}\mathbf{c}_P e^{-(\mathbf{A}_P+\mathbf{I})d}\mathbf{b}_P - \frac{e^{\omega\tau}}{S}\mathbf{c}_P(\mathbf{A}_P+\mathbf{I})e^{-(\mathbf{A}_P+\mathbf{I})d}\mathbf{b}_P \\ &+ \frac{K}{S^2}\,e^{-r\tau}\,\mathbf{c}_P e^{-\mathbf{A}_P d}\mathbf{b}_P + \frac{K}{S^2}\,e^{-r\tau}\,\mathbf{c}_P\mathbf{A}_P^{-1}e^{-\mathbf{A}_P d}\mathbf{b}_P\end{aligned}$$

Discretely Monitored Lookback Options

Let the discretely monitored risk neutral price price process be

$$S(T) = S(t)e^{\sum_{i=0}^{N} Y_i} \qquad , \qquad \mathbb{E}_{\mathbb{Q}}[e^{Y_i}] = e^{r\delta t} \tag{23}$$

where $Y_i \sim \; 2 - EPT(\mathbf{A}_{N_1}, \mathbf{b}_{N_1}, \mathbf{c}_{N_1}, \mathbf{A}_{P_1}, \mathbf{b}_{P_1}, \mathbf{c}_{P_1})$ for all $i = \{1, 2, ..., N\}$ and $Y_0 = 0$.
The minimum of the Discrete Time 2-EPT process

$$M(N) \;=\; \min_{n \in \{0,1,2,\ldots,N\}} \sum_{i=0}^{n} Y_i \tag{24}$$

has a generalised EPT distribution which can be computed via

$$M(T) \;=\; \min\{0\;,\; X_1 \;+\; \min\{0,\; X_2 \;+\; \min\{0,\; X_3 \;+\; ...\;\}\}\} \tag{25}$$

An equality in distribution ($\overset{d}{=}$) for $M(T+1)$ in terms of $M(T)$ is defined as

$$M(T+1) \overset{d}{=} \min\{0,\; X_{T+1} \;+\; M(T)\} \tag{26}$$

Discretely Monitored Lookback Options with Fixed Strike

$M(T)$:=Density of Minimum of Discrete Time 2-EPT Process of Length T

$$M(T) \sim EPT(\mathbf{A}_M, \mathbf{b}_M, \mathbf{c}_M, \mathbf{d}_M)$$

Lookback Option with Fixed Strike T

$$L(S(T), T; T, K) \ = \ \max\{K - S_{\min}(T), 0\} \tag{27}$$

Risk Neutral Option Price at time 0 with $d \leq 0$

$$L(S(0), 0; T, K) \ = \ Ke^{-rT}\mathbf{c}_M\mathbf{A}_M^{-1}e^{\mathbf{A}_M d}\mathbf{b}_M \ - \ S(0)e^{-rT}\mathbf{c}_M(\mathbf{A}_M + \mathbf{I})^{-1}e^{(\mathbf{A}_M+\mathbf{I})d}\mathbf{b}_M$$

where $d = \log(K/S(0))$

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Thank You

Questions

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